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ROSSITER'S
KEY TO
TRIGONOMETRY.

KEY TO THE EXERCISES

IN THE

ELEMENTS

OF

PLANE AND SPHERICAL TRIGONOMETRY.

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BY

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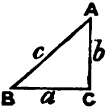
KEY

TO THE

ELEMENTS OF TRIGONOMETRY.

CHAPTER II. (Page 40.)



(1.) $$a^2 = c^2 - b^2 = 196 - 49,$$
$$= 147;$$
$$\therefore a = 12{\cdot}1.$$

Also, $$B : C :: \frac{1}{2} : 1.$$

But $\frac{1}{2}$ = *sine* of 30°; $\therefore B = 30°$.

And $A = 180° - 120° = 60°$.

Therefore $A = 60°$, $B = 30°$, and $C = 90°$.

Also, $a = 12{\cdot}1$, $b = 7$, $c = 14$,
and the triangle is completely known.

In this particular example it would be seen at once that $\frac{b}{c} = \frac{7}{14} = \frac{1}{2}$; $\therefore$ B = 30°. But the example is worked out here more generally as a guide for less obvious cases.

(2.) $\frac{b}{a} = 1$; $\therefore$ B = 45°, also A = 45°.

Also, $c^2 = a^2 + b^2 = 32$;

$\therefore c = 5{\cdot}6$.

Therefore, A = 45°, B = 45°, and C = 90°.

Also, $a = 4$, $b = 4$, and $c = 5{\cdot}6$.

(3.) $\frac{a}{c} = \frac{3}{6} = \frac{1}{2} =$ *cosine* of 60°; $\therefore$ B = 60°.

Also, $b = \sqrt{c^2 - a^2}$,

$= \sqrt{36 - 9}$,

$= 5{\cdot}2$.

Therefore, A = 30°, B = 60°, and c = 90°.

Also, $a = 3$, $b = 5{\cdot}2$, and $c = 6$.

(4.) $\frac{b}{a} = \frac{1{\cdot}74}{3} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} =$ *tangent* of 30°;

$\therefore$ B = 30°.

Also, $c = \sqrt{a^2 + b^2} = \sqrt{12}$,

$= 3{\cdot}4$.

Therefore, A = 60°, B = 30°, and C = 90°.

Also, $a = 3$, $b = 1{\cdot}74$, and $c = 3{\cdot}4$.

(5.) $\frac{a}{c} = \frac{3{\cdot}48}{4} = \frac{1{\cdot}74}{2} = \frac{\sqrt{3}}{2} =$ *cosine* 30°;

$\therefore$ B = 30°.

Also, $b = \sqrt{c^2 - a^2} = \sqrt{4} = 2.$

Therefore, A = 60°, B = 30°, and C = 90°.

Also, $a = 3{\cdot}48$, $b = 2$, and $c = 4$.

We could also find b thus:—

$$B = 30°;\ \therefore \frac{b}{c} = \textit{sine}\ 30;$$

$$\therefore \frac{b}{c} = \frac{1}{2};\ \therefore b = \frac{c}{2} = \frac{4}{2} = 2.$$

(6.) $$\frac{a}{c} = \frac{348}{400} = \frac{3{\cdot}48}{4}.$$

This problem then becomes identical with (5), except that $b = 200$ instead of 2. The triangles in (5) and (6) are similar, and therefore the angles respectively equal. But the sides in (6) are each 100 times as long as the corresponding sides in (5).

(7.) Let A C be the side of the house, B C the ground, and A B the ladder.

Then, $c = 14{\cdot}1$, A = 45°, and C = 90°.

Therefore, B also = 45°,

and $$\frac{b}{a} = \textit{tangent}\ 45° = 1;$$

$$\therefore b = a.$$

But $$b^2 + a^2 = c^2 = 200;$$

$$\therefore a^2 \text{ and } b^2 \text{ each} = 100,$$

and $$a = b = 10.$$

Therefore, A = 45°, B = 45°, and C = 90°.

Also, $a = 10$, $b = 10$, and $c = 14{\cdot}1$.

(8.) Let A C be the tree, B C the ground, and A B the string. Then $a = 10$, $c = 20$, and C = 90°.

Then, $b = \sqrt{c^2-a^2} = \sqrt{300} = 17 \cdot 4.$

And $\quad sine\ B = \frac{b}{c} = \frac{17 \cdot 4}{20} = \frac{1 \cdot 74}{2},$

$$= \frac{\sqrt{3}}{2} = sine\ 60° ;$$

$$\therefore B = 60°, \text{ and } A = 30°.$$

Therefore, A = 30°, B = 60°, and C = 90°.
Also, $a = 10$, $b = 17 \cdot 4$, and C = 20.

(9.) Let B C be the direct route across, A B the actual route, and A C the distance the boat drifts. Then, $a = 100$, B = 30°, C = 90°.

Then $\quad \frac{a}{b} = tangent\ 60° = \sqrt{3};$

$$\therefore b = \frac{100}{\sqrt{3}} = 57 \cdot 7,$$

and $\quad \frac{b}{c} = cosine\ 60° = \frac{1}{2};$

$$\therefore c = 2\,b = 115.$$

Therefore, A = 60°, B = 30°, C = 90°.
Also, $a = 100$, $b = 57 \cdot 7$, $c = 115$.

CHAPTER III. (Page 44.)

(1.) $Sin \quad = \frac{1}{3}.$

$Cosec = \frac{1}{sin} = 3.$

$Cos \quad = \sqrt{1 - sin^2} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{3}.$

$$\textit{Sec} = \frac{1}{cos} = \frac{3}{\sqrt{8}}.$$

$$\textit{Tan} = \frac{sin}{cos} = \frac{1}{3} \div \frac{3}{8} = \frac{1}{3} \times \frac{8}{3} = \frac{8}{9}.$$

$$\textit{Cot} = \frac{1}{tan} = \frac{9}{8}.$$

(2.) $$\textit{Cos} = \frac{1}{2}.$$

$$\textit{Sec} = \frac{1}{cos} = 2.$$

$$\textit{Sin} = \sqrt{1 - cos^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

$$\textit{Cosec} = \frac{1}{sin} = \frac{2}{\sqrt{3}}.$$

$$\textit{Tan} = \frac{sin}{cos} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

$$\textit{Cot} = \frac{1}{tan} = \frac{1}{\sqrt{3}}.$$

(3.) $$\textit{Tan} = 4.$$

$$\textit{Cot} = \frac{1}{tan} = \frac{1}{4}.$$

$$\textit{Sec} = \sqrt{1 + tan^2} = \sqrt{1 + 16} = \sqrt{17}.$$

$$\textit{Cos} = \frac{1}{sec} = \frac{1}{\sqrt{17}}.$$

$$\textit{Sin} = \sqrt{1 - cos^2} = \sqrt{1 - \frac{1}{17}} = \sqrt{\frac{16}{17}} = \frac{4}{\sqrt{17}}.$$

$$\textit{Cosec} = \frac{1}{sin} = \frac{\sqrt{17}}{4}.$$

(4.) $$Sin = \frac{1}{9}.$$

$$Cosec = \frac{1}{sin} = 9.$$

$$Cos = \sqrt{1 - sin^2} = \sqrt{1 - \frac{1}{81}} = \frac{\sqrt{80}}{9}.$$

$$Sec = \frac{1}{cos} = \frac{9}{\sqrt{80}}.$$

$$Tan = \frac{sin}{cos} = \frac{\frac{1}{9}}{\frac{\sqrt{80}}{9}} = \frac{1}{\sqrt{80}}.$$

$$Cot = \frac{1}{tan} = \sqrt{80}.$$

(5.) $$Cosec = 2.$$

$$Sin = \frac{1}{cosec} = \frac{1}{2}.$$

$$Cos = \sqrt{1 - sin^2} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

$$Sec = \frac{1}{cos} = \frac{2}{\sqrt{3}}.$$

$$Tan = \frac{sin}{cos} = \frac{1}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}.$$

$$Cot = \frac{1}{tan} = \sqrt{3}.$$

(6.) $$Cot = \frac{1}{4}.$$

$$Tan = \frac{1}{cot} = 4.$$

$$\textit{Sec} = \sqrt{1 + tan^2} = \sqrt{1 + 16} = \sqrt{17}$$

$$\textit{Cos} = \frac{1}{sec} = \frac{1}{\sqrt{17}}.$$

$$\textit{Sin} = \sqrt{1 - cos^2} = \sqrt{1 - \frac{1}{17}} = \frac{4}{\sqrt{17}}.$$

$$\textit{Cosec} = \frac{1}{sin} = \frac{\sqrt{17}}{4}.$$

(7.)
$$\textit{Sec} = 2.$$

$$\textit{Cos} = \frac{1}{sec} = \frac{1}{2}.$$

$$\textit{Sin} = \sqrt{1 - cos^2} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

$$\textit{Cosec} = \frac{1}{sin} = \frac{2}{\sqrt{3}}.$$

$$\textit{Tan} = \frac{sin}{cos} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}.$$

$$\textit{Cot} = \frac{1}{tan} = \frac{1}{\sqrt{3}}.$$

(8.)
$$\textit{Sin} = -\frac{1}{2}.$$

$$\textit{Cosec} = \frac{1}{sin} = -2.$$

$$\textit{Cos} = \sqrt{1 - sin^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}.$$

$$\textit{Sec} = \frac{1}{cos} = \frac{2}{\sqrt{5}}.$$

$$\textit{Tan} = \frac{sin}{cos} = \frac{-\frac{1}{2}}{\frac{\sqrt{5}}{2}} = \frac{-1}{\sqrt{5}}.$$

$$\mathit{Cot} = \frac{1}{\mathit{tan}} = -\sqrt{5}.$$

(9.) $$\mathit{Cos} = -\frac{1}{4}.$$

$$\mathit{Sec} = \frac{1}{\mathit{cos}} = -4.$$

$$\mathit{Sin} = \sqrt{1-\mathit{cos}^2} = \sqrt{1+\frac{1}{16}} = \frac{\sqrt{17}}{4}.$$

$$\mathit{Cosec} = \frac{1}{\mathit{sin}} = \frac{4}{\sqrt{17}}.$$

$$\mathit{Tan} = \frac{\mathit{sin}}{\mathit{cos}} = \frac{\sqrt{17}}{4} \div \frac{-1}{4} = -\sqrt{17}.$$

$$\mathit{Cot} = \frac{1}{\mathit{tan}} = \frac{-1}{\sqrt{17}}.$$

(10.) $$\mathit{Tan} = \frac{-1}{2}.$$

$$\mathit{Cot} = \frac{1}{\mathit{tan}} = -2.$$

$$\mathit{Sec} = \sqrt{1+\mathit{tan}^2} = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}.$$

$$\mathit{Cos} = \frac{1}{\mathit{sec}} = \frac{}{\sqrt{5}}.$$

$$\mathit{Sin} = \sqrt{1-\mathit{cos}^2} = \sqrt{1-\frac{4}{5}} = \frac{1}{\sqrt{5}}.$$

$$\mathit{Cosec} = \frac{1}{\mathit{sin}} = \sqrt{5}.$$

(11.) $$\mathit{Sec} = -7.$$

$$= \frac{1}{\mathit{sec}} = -\frac{1}{7}.$$

$$\textit{Sin} \quad = \sqrt{1 - \cos^2} = \sqrt{1 + \frac{1}{7}} = \sqrt{\frac{8}{7}}.$$

$$\textit{Cosec} \quad = \frac{1}{\sin} = \sqrt{\frac{7}{8}}.$$

$$\textit{Tan} \quad = \frac{\sin}{\cos} = \sqrt{\frac{8}{7}} \div \frac{-1}{7} = \frac{7\sqrt{8}}{\sqrt{7}} = \sqrt{56}.$$

$$\textit{Cot} \quad = \frac{1}{\tan} = \frac{1}{\sqrt{56}}.$$

(12.) *Cosec* $= -6$.

$$\textit{Sin} \quad = \frac{1}{\textit{cosec}} = \frac{-1}{6}.$$

$$\textit{Cos} \quad = \sqrt{1 - \sin^2} = \sqrt{1 - \frac{1}{36}} = \sqrt{\frac{35}{36}} = \frac{\sqrt{35}}{6}.$$

$$\textit{Sec} \quad = \frac{1}{\cos} = \frac{6}{\sqrt{35}}.$$

$$\textit{Tan} \quad = \frac{\sin}{\cos} = \frac{-1}{6} \div \frac{\sqrt{35}}{6} = -\frac{1}{\sqrt{35}}.$$

$$\textit{Cot} \quad = \frac{1}{\tan} = \sqrt{35}.$$

(13.) If *tan* = 7·784, find *sin* and *sec*.

$$\textit{Sin} = \frac{\tan}{\sqrt{\tan^2 + 1}},$$

$$= \frac{7 \cdot 784}{\sqrt{7 \cdot 784^2 + 1}},$$

$$= \frac{7 \cdot 784}{7 \cdot 8},$$

$$= \cdot 998.$$

$$\begin{aligned} Sec &= \sqrt{1 + tan^2}, \\ &= \sqrt{1 + 7{\cdot}784^2}, \\ &= 7{\cdot}8. \end{aligned}$$

(14.) If *sin* = ·47, find *cot* and *cos*.

$$\begin{aligned} Cot &= \frac{\sqrt{1 - sin^2}}{sin}, \\ &= \frac{\sqrt{1 - {\cdot}47^2}}{{\cdot}47}, \\ &= \frac{{\cdot}88}{{\cdot}47}, \\ &= 1{\cdot}87. \end{aligned}$$

$$\begin{aligned} Cos &= \sqrt{1 - sin^2}, \\ &= \sqrt{1 - {\cdot}47^2}, \\ &= \sqrt{{\cdot}7791} \\ &= {\cdot}88. \end{aligned}$$

(15.) If *tan* = 7·41, find *cos* and *cosec*.

$$\begin{aligned} Cos &= \frac{1}{\sqrt{1 + tan^2}}, \\ &= \frac{1}{\sqrt{1 + 7{\cdot}41^2}}, \\ &= \frac{1}{7{\cdot}47}, \\ &= {\cdot}133. \end{aligned}$$

$$Cosec = \frac{\sqrt{tan^2 + 1}}{tan},$$

$$= \frac{\sqrt{1+7{\cdot}41^2}}{tan},$$

$$= \frac{7{\cdot}47}{7{\cdot}41},$$

$$= 1{\cdot}008.$$

(16.) If *sec* = 4·8, find *sin* and *tan*.

$$Sin = \frac{\sqrt{sec^2 - 1}}{sec},$$

$$= \frac{\sqrt{4{\cdot}8^2 - 1}}{4{\cdot}8},$$

$$= \frac{4{\cdot}69}{4{\cdot}8},$$

$$= {\cdot}97.$$

$$Tan = \sqrt{sec^2 - 1},$$

$$= \sqrt{4{\cdot}8^2 - 1},$$

$$= 4{\cdot}69.$$

(17.) If *cos* = ·57, find *tan* and *sec*.

$$Tan = \frac{\sqrt{1 - cos^2}}{cos},$$

$$= \frac{\sqrt{1 - {\cdot}57^2}}{{\cdot}57}$$

$$= \frac{{\cdot}82}{{\cdot}57},$$

$$= 1{\cdot}43.$$

$$Sec = \frac{1}{cos},$$
$$= \frac{1}{\cdot 57},$$
$$= 1\cdot 7.$$

(18.) If *cosec* = 4·84, find *tan* and *cos*.

$$Tan = \frac{1}{\sqrt{cosec^2 - 1}},$$
$$= \frac{1}{\sqrt{4\cdot 84^2 - 1}},$$
$$= \frac{1}{4\cdot 73},$$
$$= \cdot 21.$$

$$Cos = \frac{\sqrt{cos^2 - 1}}{cosec},$$
$$= \frac{4\cdot 73}{4\cdot 84},$$
$$= \cdot 97.$$

(19.) If *sec* = 3·6, find *sin* and *tan*.

$$Sin = \frac{\sqrt{ec^2 = 1}}{sec},$$
$$= \frac{\sqrt{3\cdot 6^2 - 1}}{3\cdot 6},$$
$$= \frac{3\cdot 4}{3\cdot 6},$$
$$= \cdot 94.$$

$$Tan = \sqrt{sec^2 - 1},$$
$$= \sqrt{3{\cdot}6^2 - 1},$$
$$= 3{\cdot}4.$$

(20.) If *cot* = 7·4, find all the other ratios.

$$Sin = \frac{1}{\sqrt{1 + cot^2}},$$
$$= \frac{1}{\sqrt{1 + 7{\cdot}4^2}},$$
$$= \frac{1}{7{\cdot}46},$$
$$= {\cdot}13.$$

$$Cos = \frac{cot}{\sqrt{1 + cot^2}},$$
$$= \frac{7{\cdot}4}{\sqrt{1 + 7{\cdot}4^2}},$$
$$= \frac{7{\cdot}4}{7{\cdot}46},$$
$$= {\cdot}991.$$

$$Tan = \frac{1}{cot},$$
$$= \frac{1}{7{\cdot}4},$$
$$= {\cdot}135.$$

$$Sec = \frac{1}{cos},$$

$$= \frac{1}{\cdot 991},$$

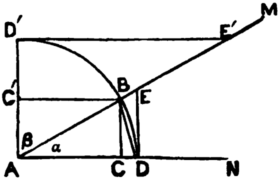
$$= \cdot 1009.$$

$$Cosec = \frac{1}{sin},$$

$$= \frac{1}{\cdot 13},$$

$$= 7 \cdot 6.$$

CHAPTER IV. (Page 51.)



(1.) AB = 4, AE = 5, BC = 1. Find AC, AD, and DE.

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{16 - 1} = \sqrt{15}.$$

Also, $AD : AC :: AE : AB :: 5 : 4;$

$\therefore AD : \sqrt{15} :: 5 : 4;$ $\therefore 4\,AD = 5\sqrt{15};$

$$\therefore AD = \frac{5}{4}\sqrt{15}.$$

And $DE : BC :: AE : AB :: 5 : 4;$

$$\therefore DE : 1 :: 5 : 4;$$

$$\therefore DE = \frac{5}{4}.$$

Therefore $AC = \sqrt{15}$, $AD = \frac{5}{4}\sqrt{15}$, and $DE = \frac{5}{4}$.

(2.) $AB=5$, $CD=1$. Find AC, BE, BC, and DE.

$$AD = AB; \quad \because \text{ both are radii};$$

$$\therefore AC = 4; \quad \therefore BC = \sqrt{25-16} = 3.$$

Also, $$BE : AB :: CD : AC;$$

$$\therefore BE : 5 :: 1 : 4;$$

$$\therefore BE = \frac{5}{4}.$$

And $$DE : BC :: AE : AB;$$

$$\therefore DE : 3 :: \frac{25}{4} : 5,$$

and $$DE = \frac{15}{4}.$$

Therefore $AC=4$, $BE=\frac{5}{4}$, $BC=3$, and $DE=\frac{15}{4}$.

(3.) $AC = DC$. Find the value of $\frac{AB}{AE}$ and of $\frac{BC}{DE}$.

$$AD = 2AC; \quad \therefore AB = 2AC;$$

$$\therefore AE = 2AB,$$

and $$\frac{AB}{AE} = \frac{1}{2}.$$

But also $$BC : DE :: AB : AE;$$

$$\therefore \frac{BC}{DE} = \frac{AB}{AE} = \frac{1}{2}.$$

In this example the secant = twice the radius; also, the tangent = twice the sine, and the cosine $= \frac{\text{radius}}{2} = \frac{\text{secant}}{4}$. The angle of which these are true = 60°.

(4.) DE is *always* greater than BC, except when both $= 0$; *i.e.*, when angle $\alpha = 0$. In this case, angle $\beta = 90$.

(5.) Find the magnitude of α and β, when D′ E′ = D E.

$$DE = \text{tangent of } \alpha,\ D'E' = \text{tangent of } \beta.$$

These can only be equal when $\alpha = \beta$; *i. e.*, when each = 45°.

(6.) Find the magnitude of angle A B C, when A E′ = 2 A D′.

$$AE' = 2AD' \therefore AB = 2BC;$$

$$\therefore \frac{BC}{AB} = \frac{1}{2} = \sin 30°.$$

$$\therefore \text{angle } \alpha = 30°.$$

$$\text{But angle } ABC = 90° - 30° = 60°.$$

(7.) Prove $\beta = 30°$ when A B = B D.

$$AB = BD, \text{ also, } AB = AD;$$

$$\therefore BD = AD;$$

$$\therefore \text{the triangle } ABD \text{ is equilateral};$$

$$\therefore \text{each of its angles is } 60°;$$

$$\therefore \text{angle } \alpha = 60°;$$

$$\therefore \text{angle } \beta = 30°.$$

(8.) A B = 5. Find A C, A E, and D E, if $\beta = 30°$;

$$\therefore \beta = 30°;\ \therefore \alpha = 60°;$$

$$\text{cosine of } 60° = \frac{1}{2} = \frac{AC}{AB};$$

$$\therefore AC = \frac{AB}{2} = \frac{5}{2}.$$

$$\text{Also, } AE = \text{secant of } 60° = \frac{1}{\cos} = 2;$$

$$\therefore \frac{AE}{AB} = 2 \therefore AE = 10.$$

And $DE = \sqrt{AE^2 - AD^2} = \sqrt{100 - 25} = \sqrt{75}$.

Therefore, $AC = \frac{5}{2}$, $AE = 10$, and $DE = \sqrt{75}$.

(9.) If *sin* $a = \cdot 484$, find *cos*, *tan*, and *sec* of β.

$$Sin\ a = cos\ \beta;$$
$$\therefore cos\ \beta = \cdot 484.$$

$$Tan\ \beta = \frac{\sqrt{1 - cos^2}}{cos},$$
$$= \frac{\sqrt{1 - \cdot 484^2}}{\cdot 484},$$
$$= \frac{\cdot 874}{\cdot 484},$$
$$= 1\cdot 8.$$

$$Sec\ \beta = \frac{1}{cos},$$
$$= \frac{1}{\cdot 484},$$
$$= 2\cdot 07.$$

(10.) If *cos* $a = \cdot 574$, find *tan* and *cosec* of β.

$$Cos\ a = sin\ \beta;$$
$$\therefore sin\ \beta = \cdot 574.$$

$$Tan\ \beta = \frac{sin\ \beta}{\sqrt{1 - sin^2 \beta}},$$
$$= \frac{\cdot 574}{\sqrt{1 - \cdot 574^2}},$$

$$\begin{aligned} &= \frac{\cdot 574}{\cdot 818}, \\ &= \cdot 701. \end{aligned}$$

$$\begin{aligned} Cosec &= \frac{1}{sin}, \\ &= \frac{1}{\cdot 574}, \\ &= 1\cdot 74. \end{aligned}$$

(11.) If *tan* $\alpha = 4\cdot 79$, find *sin* and *cos* of β.

$$\begin{aligned} Tan\ \alpha &= cot\ \beta; \\ \therefore cot\ \beta &= 4\cdot 79. \end{aligned}$$

$$\begin{aligned} Sin\ \beta &= \frac{1}{\sqrt{1 + cot^2\ \beta}}, \\ &= \frac{1}{\sqrt{1 + 4\cdot 79^2}}, \\ &= \frac{1}{4\cdot 89}, \\ &= \cdot 204 \end{aligned}$$

$$\begin{aligned} Cos\ \beta &= \sqrt{1 - sin^2}, \\ &= \sqrt{1 - \cdot 204^2}, \\ &= \cdot 979. \end{aligned}$$

(12.) If *cot* $\alpha = 6\cdot 84$, find *sec* and *sin* of β.

$$\begin{aligned} Cot\ \alpha &= tan\ \beta; \\ \therefore tan\ \beta &= 6\cdot 84. \end{aligned}$$

$$\begin{aligned} Sec\ \beta &= \sqrt{1 + tan^2\beta}, \\ &= \sqrt{1 + 6 \cdot 84^2}, \\ &= 6 \cdot 91. \end{aligned}$$

$$\begin{aligned} Sin\ \beta &= \frac{tan\ \beta}{\sqrt{tan^2\beta + 1}}, \\ &= \frac{6 \cdot 84}{\sqrt{6 \cdot 84 + 1}}, \\ &= \frac{6 \cdot 84}{6 \cdot 91}, \\ &= \cdot 989. \end{aligned}$$

(13.) If $sec\ \alpha = 7 \cdot 9$, find *tan* and *cosec* of β.

$$Sec\ \alpha = cosec\ \beta;$$

$$\therefore\ cosec\ \beta = 7 \cdot 9.$$

$$\begin{aligned} Tan\ \beta &= \frac{1}{\sqrt{cosec^2 - 1}}, \\ &= \frac{1}{\sqrt{79^2 - 1}}, \\ &= \frac{1}{7 \cdot 8}, \\ &= \cdot 128. \end{aligned}$$

(14.) If $cosec\ \alpha = 2 \cdot 57$, find all the ratios of β.

$$Cosec\ \alpha = sec\ \beta;$$

$$\therefore\ sec\ \beta = 2 \cdot 57.$$

$$Cos\ \beta = \frac{1}{sec\ \beta},$$
$$= \frac{1}{2{\cdot}57},$$
$$= {\cdot}38.$$

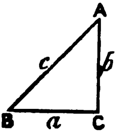
$$Sin = \sqrt{1 - cos^2}$$
$$= \sqrt{1 - {\cdot}38^2},$$
$$= {\cdot}925.$$

$$Cosec = \frac{1}{sin},$$
$$= \frac{1}{{\cdot}925},$$
$$= {\cdot}107.$$

$$Tan = \sqrt{sec^2 - 1},$$
$$= \sqrt{2{\cdot}57^2 - 1},$$
$$= 2{\cdot}36.$$

$$Cot = \frac{1}{tan},$$
$$= \frac{1}{2{\cdot}36},$$
$$= {\cdot}42.$$

CHAPTER V. (Page 59.)



(1.) Tangent of B $= \frac{b}{a} = \frac{4}{3}$;

.·. B is between 45° and 60°.

For *tan* 45° = 1, and *tan* 60° = 1·7.

But *b* being $-b$, .·. it is drawn below B C; and *a* being +, .·. the angle B is in the 4th quadrant; .·. it is 270° greater than if in the 1st quadrant; .·. B is between 315° and 330°.

And A = between 90° − 60° and 90° − 45°,
= between 30° and 45°.

(2.) This is precisely as in last example, except that *a* and *b* being both −, the angle must be in the 3rd quadrant;

.·. B = between 225° and 240°,
and A = between 30° and 45°.

(3.) The *sine* of an angle $= \frac{1}{2}$, the *cosine* $= \frac{-\sqrt{3}}{2}$

Find magnitude of angle, and values of remaining ratios.

The *sine* is + between 0° and 180°.

The *cosine* is − between 90° and 270°.

.·. the given angle is between 90° and 180°.

But $\frac{1}{2}$ = *sine* of 30° = *sine* of 90° + 30°.

$\therefore$ the given angle = 120°,
and is in the 2nd quadrant.

Also, $\quad sin = \frac{1}{2}; \therefore cosec = 2;$

$$cos = \frac{-\sqrt{3}}{2}; \therefore sec = \frac{-2}{\sqrt{3}};$$

$$tan = \frac{-1}{\sqrt{3}}; \therefore cot = -\sqrt{3}.$$

(4.) The *secant* = 2, *tangent* = $-\sqrt{3}$.
Find other ratios, and value of angle.
The *secant* is + in 1st and 4th quadrant.
The *tangent* is − in 2nd and 4th quadrant.
$\therefore$ the given angle is in the 4th quadrant.
But 2 = *secant* of 60° = *secant* of 270° + 60°;
$\therefore$ the given angle is 330°.

Then also, $sec = 2; \quad \therefore cos = \frac{1}{2};$

$$tan = -\sqrt{3}; \therefore cot = \frac{-1}{\sqrt{3}};$$

also, $\quad sin = \frac{-\sqrt{3}}{2}; \therefore cosec = \frac{-2}{\sqrt{3}}.$

(5.) The *cotangent* = − 2 *cosine.*
Find the other ratios, and the angle.

Now $\frac{a}{b} = 2\frac{a}{c}; \therefore 2\frac{b}{a} = \frac{c}{a}; \therefore 2b = c;$

$$\therefore \frac{b}{c} = \frac{1}{2} = \textit{sine of } B = \textit{sine } 30°;$$

$$\therefore B = 30°.$$

The ratios of 30° are already known:

$$\begin{aligned} &sine = \frac{1}{2}, && cosecant = 2, \\ &cosine = \frac{\sqrt{3}}{2}, && secant = \frac{2}{\sqrt{3}}, \\ &tangent = \frac{1}{\sqrt{3}}, && cotangent = \sqrt{3}. \end{aligned}$$

Now as to the quadrant in which the angle is:

The *cotangent* may be positive and the *cosine* negative, or *vice versâ.* Now *cot* and *cos* are both alike between 0° and 180°, but unlike between 180° and 360°;

$$\therefore \text{ the angle may be } 180° + 30° = 210°;$$
$$\text{or, } 270° + 30° = 300.$$

(6.) The *sine* $= \frac{tangent}{2}$.

Find the other ratios and the angle.

Now, $$2\frac{b}{c} = \frac{b}{a}; \quad \therefore c = 2a.$$

$$\therefore \frac{a}{c} = \frac{1}{2} = cos \text{ of } 60°.$$

$$\therefore B = 60°.$$

The ratios are already known:

$$\begin{aligned} &sin = \frac{\sqrt{3}}{2}, && cosec = \frac{2}{\sqrt{3}}, \\ &cos = \frac{1}{2}, && sec = 2. \\ &tan = \sqrt{3}, && cot = \frac{1}{\sqrt{3}}. \end{aligned}$$

As to the quadrant in which the angle is, it must be either the 1st or 4th, since the *sine* and *tangent* have the same sign, which is not the case in the 2nd and 3rd quadrants.

Therefore, the angle may be 60° or 270° + 60° = 330°.

(7.) Find all the ratios of 240°.

The angle 240° is greater than 180°, but less than 270°, therefore the radius of the circle will be the third quadrant, and the ratios of 240° are the same as those of 240° − 180° = 60°, excepting that some of them will be minus. In fact, four out of the six will be −, the *tan* and *cot* only being +.

Thus, if the angle 60°,

then, $$\begin{aligned} sin &= \frac{\sqrt{3}}{2}, \text{ (page 35)} \\ cos &= \frac{1}{2}, \\ tan &= \sqrt{3}, \\ sec &= 2, \\ cosec &= \frac{2}{\sqrt{3}}, \\ cot &= \frac{1}{\sqrt{3}}. \end{aligned}$$

But, if the angle be 240° (*i. e.*, 180° + 60°),

then, $$\begin{aligned} sin &= -\frac{\sqrt{3}}{2}, \\ cos &= -\frac{1}{2}, \\ tan &= \sqrt{3}, \\ sec &= -2, \\ cosec &= -\frac{2}{\sqrt{3}}, \\ cot &= \frac{1}{\sqrt{3}}. \end{aligned}$$

(8.) Find the *sin* and *cot* of 375°.

The ratios of 375° are the same as those of 360° + 15° (*i. e.,* the angle is greater than four right angles), and the radius will be in the first quadrant, so that the signs will be the same as for 15°. In fact, the ratios of 15° and 360° + 15° are identical.

Therefore, if the angle be 15° or 375°,

$$sin = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$cos = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

$$tan = \frac{\sqrt{3}-1}{\sqrt{3}+1},$$

$$sec = \frac{2\sqrt{2}}{\sqrt{3}+1},$$

$$cosec = \frac{2\sqrt{2}}{\sqrt{3}-1},$$

$$cot = \frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

A collection of these values is given on page 155.

(9.) Find the *tan* and *sec* of 285°.

The angle 285° may be written 270° + 15°, or as 360° − 75°, and therefore the ratios will be the same − 75° = the ratios of + 75°, except that they are all − except the *cosine* and *secant.*

Then, if the angle be 75°, (page 35)

$$sin = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

$$cos = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$tan = \frac{\surd 3 + 1}{\surd 3 - 1},$$

$$cot = \frac{\surd 3 - 1}{\surd 3 + 1},$$

$$sec = \frac{2\surd 2}{\surd 3 - 1},$$

$$cosec = \frac{2\surd 2}{\surd 3 + 1}.$$

But if the angle be − 75°,

$$sin = \frac{\surd 3 + 1}{2\surd 2},$$

$$cos = \frac{\surd 3 - 1}{2\surd 2},$$

$$tan = \frac{\surd 3 + 1}{\surd 3 - 1},$$

$$cot = \frac{\surd 3 - 1}{\surd 3 + 1},$$

$$sec = \frac{2\surd 2}{\surd 3 - 1},$$

$$cosec = \frac{\surd 2\surd 2}{\surd 3 + 1}.$$

(10.) Find the *cot*, *sec*, and *sin* of 165°.

An angle of 165° may be written 180° − 15°, and the ratios of 165° are therefore the same as those of 15°, excepting that the *sine* and *cosecant* only are +, the other being −.

$$\therefore\ sin\ 165° = \ sin\ 15° = -\frac{\surd 3 - 1}{2\surd 2},$$

$$cos\ \text{,,} = -\,cos\ \text{,,} = -\frac{\surd 3 + 1}{2\surd 2},$$

$$\tan\ \text{„} = -\tan\ 15^\circ = -\frac{\sqrt{3}-1}{\sqrt{3}+1},$$

$$\sec\ \text{„} = -\sec\ \text{„} = -\frac{2\sqrt{2}}{\sqrt{3}+1},$$

$$\operatorname{cosec}\ \text{„} = \operatorname{cosec}\ \text{„} = -\frac{2\sqrt{2}}{\sqrt{3}-1},$$

$$\cot\ \text{„} = \cot\ \text{„} = -\frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

(11.) Find all the ratios of 300°.

$$300^\circ = 360^\circ - 60^\circ;$$

∴ ratios of 300° = ratios of 60°,
excepting that *sec* and *cos* only are + ;

$$\therefore \sin\ 300^\circ = -\sin\ 60^\circ = -\frac{\sqrt{3}}{2},$$

$$\cos\ \text{„} = \cos\ \text{„} = -\frac{1}{2},$$

$$\tan\ \text{„} = -\tan\ \text{„} = -\sqrt{3},$$

$$\sec\ \text{„} = \sec\ \text{„} = -2,$$

$$\operatorname{cosec}\ \text{„} = -\operatorname{cosec}\ \text{„} = -\frac{2}{\sqrt{3}},$$

$$\cot\ \text{„} = -\cot\ \text{„} = -\frac{1}{\sqrt{3}}.$$

(12.) Find all the ratios of 120°.

$$120^\circ = 180^\circ - 60^\circ;$$

∴ ratios of 120° = ratios of 60°,
excepting that *sin* and *cosec* only are + ;

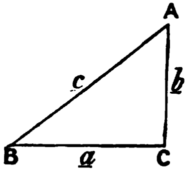
$$\therefore \sin\ 120^\circ = \sin\ 60^\circ = \frac{\sqrt{3}}{2},$$

$$\cos\ \text{„} = -\cos\ 60^\circ = -\frac{1}{2},$$

$$\begin{array}{llllll} \tan & ,, & = -\tan & ,, & = & -\sqrt{3}, \\ \sec & ,, & = -\sec & ,, & = & -\ 2, \\ \mathrm{cosec} & ,, & = \ \ \mathrm{cosec} & ,, & = & -\dfrac{2}{\sqrt{3}}. \\ \cot & ,, & = -\cot & ,, & = & -\dfrac{1}{\sqrt{3}}. \end{array}$$

Notice, that from these two it follows that the ratios of 300° are the same as those of 120°, excepting the signs. That is, ratios of $180° + 60°$ = ratios of $180° - 60°$. Also, ratios of 60° = ratios of $180° \pm 60°$, and generally, ratios of $x°$ = ratios of $180° \pm x°$.

CHAPTER VI. (Page 64.)



RIGHT-ANGLED TRIANGLES.

(1.) $\dfrac{b}{c} = \dfrac{2}{4} = \dfrac{1}{2} = \sin 30°$

$\therefore$ B = 30°.

Also, A = 60°.

And $a = \sqrt{c^2 - b^2} = \sqrt{12}.$

(2.) $a = 17 \cdot 4$, $c = 20$. Find A, B, and b.

$$\frac{a}{c} = \frac{17 \cdot 4}{20} = \frac{1 \cdot 74}{2} = \frac{\sqrt{3}}{2} = \sin 60°;$$

$$\therefore\ A = 60°, \text{ and } B = 30°.$$

Also, $$b = \sqrt{c^2 - a^2} = \sqrt{100} = 10;$$

or, $$\frac{b}{c} = \sin B = \sin 30°;$$

$$\therefore\ b = \frac{c}{2} = \frac{20}{2} = 10.$$

The triangle in this example is similar to the one in (1), but having the sides each five times as great.

(3.) $$c^2 = a^2 + b^2,$$

$$\therefore\ 8 = a^2 + 4,$$

and $$a = 2;$$

$$\therefore\ a = b.$$

Now, $$\frac{b}{a} = 1 = \tan 45°;$$

$$\therefore\ B = 45°.$$

Also, $$A = 90° - 45° = 45°.$$

$$\therefore\ A = 45°, B = 45°, C = 90,$$

$$a = 2, \quad b = 2, \quad c = \sqrt{8}.$$

(4.) If $a = 47$ ft., $b = 39$ ft., and $C = 90°$, find A, B, and c.

$$c^2 = a^2 + b^2;$$

$$\therefore\ c = \sqrt{a^2 + b^2},$$

$$= \sqrt{3630},$$

$$= 61.$$

$$\frac{\sin A}{\sin C} = \frac{a}{c} = \frac{47}{61};$$

$$\therefore\ \sin A = \frac{a}{c} \sin C,$$

$$
\begin{aligned}
&= \frac{47}{61}, \\
&= \cdot 77049, \\
&= \sin 50^\circ\, 20'; \\
\therefore\ A &= 50^\circ\, 20'.
\end{aligned}
$$

$$
\begin{aligned}
B &= 90^\circ - A, \\
&= 90^\circ - 50^\circ\, 20', \\
&= 39^\circ\, 40'.
\end{aligned}
$$

Therefore the triangle is completely known.

(5.) If $a = 2\cdot 75$ ft., $c = 3\cdot 5$ ft., and $C = 90^\circ$, find A, B, and b.

$$
\begin{aligned}
c^2 &= a^2 + b^2; \\
\therefore\ b^2 &= c^2 - a^2,
\end{aligned}
$$

and

$$
\begin{aligned}
b &= \sqrt{c^2 - a^2}, \\
&= \sqrt{4\cdot 6875}, \\
&= 2\cdot 19.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin A}{\sin C} &= \frac{a}{c} = \frac{2\cdot 75}{3\cdot 5}; \\
\therefore\ \sin A &= \frac{2\cdot 75}{3\cdot 5} \sin C, \\
&= \frac{2\cdot 75}{3\cdot 5}, \\
&= \sin 51^\circ\, 47' - ; \\
\therefore\ A &= 51^\circ\, 47',
\end{aligned}
$$

and

$$
\begin{aligned}
B &= 90^\circ - 51^\circ\, 47', \\
&= 38^\circ\, 13',
\end{aligned}
$$

and the triangle is completely known.

(6.) If $a = 1\cdot 5$ in., $A = 14°$, and $C = 90°$, find B, b, and c.

$$\frac{Sin\ A}{Sin\ C} = \frac{a}{c};$$

$$\therefore \frac{c}{a} = \frac{sin\ C}{sin\ A};$$

$$\therefore c = \frac{a\ sin\ C}{sin\ A},$$

$$= \frac{1\cdot 5\ sin\ 90°}{sin\ 14°}$$

$$= \frac{1\cdot 5}{\cdot 24192},$$

$$= 6\cdot 2.$$

Then, $$b^2 = c^2 - a^2;$$

$$\therefore b = \sqrt{c^2 - a^2},$$

$$= \sqrt{40\cdot 69},$$

$$= 6\cdot 37.$$

And $$B = 90° - A,$$

$$= 90° - 14°,$$

$$= 76°,$$

and the triangle is completely known.

(7.) If $c = 7$ in., $A = 24°$, and $C = 90°$, find B, a, and b.

$$\frac{Sin\ A}{Sin\ C} = \frac{a}{c};$$

$$\therefore a = \frac{c\ sin\ A}{sin\ C},$$

$$= \frac{7 \times \cdot 4067366}{1},$$

$$= \cdot 4067366 + 7,$$

$$= 2\cdot 8471562.$$

Then, $$c^2 = a^2 + b^2;$$

$$\therefore b = \sqrt{c^2 - a^2}.$$

$$= 6\cdot 39.$$

Also, $$B = 90^\circ - A,$$

$$= 90^\circ - 24^\circ,$$

$$= 66^\circ.$$

(8.) If $a = 1$ mile, $b = 795$ yards, $C = 90^\circ$, find A, B, and c.

$$c^2 = a^2 + b^2;$$

$$\therefore c = \sqrt{a^2 + b^2},$$

$$= 1931.$$

Then, $$\frac{\sin A}{\sin C} = \frac{a}{c};$$

$$\therefore \sin A = \frac{1760}{1931};$$

$$\therefore A = 64^\circ\ 21'.$$

And $$B = 90^\circ - A,$$

$$= 25^\circ\ 39'.$$

(9.) If $a = \cdot 49$ of a mile, $A = 17^\circ$, $C = 90^\circ$, find B, b, and c.

$$\begin{aligned} B &= 90^\circ - A, \\ &= 73^\circ. \end{aligned}$$

$$\begin{aligned} b &= \frac{a \times \sin B}{\sin A}, \\ &= \frac{\cdot 49 \times \sin 73^\circ}{\sin 17^\circ}, \\ &= \frac{\cdot 49 \times \cdot 9563048}{\cdot 2923717}, \\ &= 1\cdot 59. \end{aligned}$$

Also,
$$\begin{aligned} c &= \sqrt{a^2 + b^2}, \\ &= \sqrt{\cdot 49^2 + 1\cdot 59^2}, \\ &= 1\cdot 65. \end{aligned}$$

(10.) If $a = 9$ in., $c = 18$ in., $C = 90$, find A, B, and b.

$$\begin{aligned} b &= \sqrt{c^2 - a^2}. \\ &= \sqrt{18^2 - 9^2}, \\ &= 15\cdot 5. \end{aligned}$$

Then,
$$\begin{aligned} \frac{\sin A}{\sin C} &= \frac{a}{c}, \\ &= \frac{9}{18}; \\ \therefore \sin A &= \frac{9}{18} \sin c, \\ &= \frac{1}{2}; \\ \therefore A &= 30^\circ. \end{aligned}$$

And
$$\begin{aligned} B &= 90^\circ - A, \\ &= 90^\circ - 30^\circ, \\ &= 60^\circ. \end{aligned}$$

(11.) If $c = 47$ yards, $A = 4^\circ$, $C = 90^\circ$, find B, a, and b.

$$\begin{aligned} B &= 90^\circ - A, \\ &= 90^\circ - 4, \\ &= 86^\circ. \end{aligned}$$

Then,
$$\begin{aligned} a &= \frac{c \times \sin A}{\sin C}, \\ &= 47 \times \cdot 0697565, \\ &= 3 \cdot 279. \end{aligned}$$

And
$$\begin{aligned} b &= \sqrt{c^2 - a^2}, \\ &= \sqrt{47^2 - 3 \cdot 279^2}, \\ &= 46 \cdot 8. \end{aligned}$$

CHAPTER VII. (Page 72.)

(1.) Find the values of the *sine* and *cosine* of 120°, in the terms of the ratios of 90° and 30°.

$$\begin{aligned} \sin 120^\circ &= \sin 90^\circ \,.\, \cos 30^\circ + \cos 90^\circ \,.\, \sin 30^\circ, \\ &= 1 \times \frac{\sqrt{3}}{2} + 0 \times \frac{1}{2}, \\ &= \frac{\sqrt{3}}{2}, \end{aligned}$$

which is the same as $\sin 60^\circ$;

i. e., $\sin 60^\circ =$ (in magnitude) $\sin 180^\circ \pm 60^\circ$.

Then,
$$\begin{aligned}\cos 120^\circ &= \cos 90^\circ . \cos 30^\circ - \sin 90^\circ . \sin 30^\circ,\\ &= 0 \times \frac{\sqrt{3}}{2} - 1 \times \frac{1}{2},\\ &= -\frac{1}{2},\end{aligned}$$

which is = *cos* 60°, but with different sign;

i. e., $\cos 60^\circ = \cos 180^\circ \pm 60^\circ$.

The other ratios are = those of 60°, with the signs proper to the second quadrant.

(2.)
$$\begin{aligned}\sin 120^\circ &= \sin 60^\circ . \cos 60^\circ + \cos 60^\circ . \sin 60^\circ,\\ &= \frac{\sqrt{3}}{2} . \frac{1}{2} + \frac{1}{2} . \frac{\sqrt{3}}{2},\\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4},\\ &= \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}, \text{ as before.}\end{aligned}$$

$$\begin{aligned}\cos 120^\circ &= \cos 60^\circ . \cos 60^\circ - \sin 60^\circ . \sin 60^\circ,\\ &= (\cos 60^\circ)^2 - (\sin 60^\circ)^2,\\ &= \frac{1}{4} - \frac{3}{4},\\ &= -\frac{1}{2}, \text{ as before.}\end{aligned}$$

Also,
$$\begin{aligned}\sin 120^\circ &= \sin (180^\circ - 60^\circ),\\ &= \sin 180^\circ . \cos 60^\circ - \cos 180^\circ . \sin 60^\circ,\\ &= 0 . \frac{1}{2} + 1 . \frac{\sqrt{3}}{2},\\ &= \frac{\sqrt{3}}{2}, \text{ as before.}\end{aligned}$$

Also, $\cos 120^{\circ} = \cos(180^{\circ} - 60^{\circ})$,

$$\begin{aligned} &= \cos 180^{\circ} . \cos 60^{\circ} + \sin 180^{\circ} . \sin 60^{\circ}, \\ &= -1 . \frac{1}{2} + 0 . \frac{\sqrt{3}}{2}, \\ &= -\frac{1}{2}, \text{ as before.} \end{aligned}$$

(3.) $\operatorname{Sin} 90^{\circ} = \sin(45^{\circ} + 45^{\circ})$,

$$\begin{aligned} &= \sin 45^{\circ} . \cos 45^{\circ} + \cos 45^{\circ} . \sin 45^{\circ}, \\ &= 2 \sin 45^{\circ} . \cos 45^{\circ}, \\ &= 2 . \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{2}}, \\ &= 1 . \text{ as before.} \end{aligned}$$

$$\begin{aligned} \operatorname{Cos} 90^{\circ} &= \cos 45^{\circ} . \cos 45^{\circ} - \sin 45^{\circ} . \sin 45^{\circ}, \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2, \\ &= 0, \text{ as before} \end{aligned}$$

$$\begin{aligned} \operatorname{Tan} 90^{\circ} &= \frac{\sin 45^{\circ} . \cos 45^{\circ} + \cos 45^{\circ} . \sin 45^{\circ}}{\cos 45^{\circ} . \cos 45^{\circ} - \sin 45^{\circ} . \sin 45^{\circ}}, \\ &= \frac{\frac{1}{\sqrt{2}} . \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} . \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{2}}}, \\ &= \frac{1}{0} = \infty, \text{ as before.} \end{aligned}$$

(4.) $\operatorname{Sin} 60^{\circ} = \sin(90^{\circ} - 30^{\circ})$,

$$= \sin 90^{\circ} . \cos 30^{\circ} + \cos 90^{\circ} . \sin 30^{\circ},$$

$$= 1\,.\,\frac{\sqrt{3}}{2} + 0\,.\,\frac{1}{2}.$$

$$= \frac{\sqrt{3}}{2}, \text{ as before.}$$

$$\textit{Cos } 60^\circ = \cos\,(90^\circ - 30^\circ),$$

$$= \cos 90^\circ\,.\,\cos 30^\circ + \sin\,90^\circ\,.\,\sin\,30^\circ,$$

$$= 0\,.\,\frac{\sqrt{3}}{2} + 1\,.\,\frac{1}{2},$$

$$= \frac{1}{2}, \text{ as before.}$$

$$\textit{Sec } 60^\circ = \sec\,(90^\circ - 30^\circ),$$

$$= \frac{1}{\cos}\,(90^\circ - 30^\circ),$$

$$= \frac{1}{0\,.\,\frac{\sqrt{3}}{2} + 1\,.\,\frac{1}{2}},$$

$$= 2, \text{ as before.}$$

(5.) $$\textit{Sine } 270^\circ = \sin\,(180^\circ + 90^\circ),$$

$$= \sin 180^\circ\,.\,\cos 90^\circ + \cos 180^\circ\,.\,\sin 90^\circ.$$

$$= 0\,.\,0 - 1\,.\,1,$$

$$= -1, \text{ as before.}$$

$$\textit{Sec } 270^\circ = \sec\,(180^\circ + 90^\circ),$$

$$= \frac{1}{\cos}(180^\circ + 90^\circ),$$

$$\begin{aligned}
&= \frac{1}{\cos 180^\circ . \cos 90^\circ - \sin 180^\circ . \sin 90^\circ}, \\
&= \frac{1}{-1 . 0 - 0 . 1}, \\
&= \infty, \text{ as before.}
\end{aligned}$$

$$\begin{aligned}
\tan 270^\circ &= \tan (180^\circ + 90^\circ), \\
&= \frac{\sin}{\cos} (180^\circ + 90^\circ), \\
&= \frac{-1}{0}, \\
&= \infty, \text{ as before.}
\end{aligned}$$

(6.)
$$\begin{aligned}
\cos 180^\circ &= \cos (90^\circ + 90^\circ), \\
&= \cos 90^\circ . \cos 90^\circ - \sin 90^\circ . \sin 90^\circ, \\
&= 0^2 - 1^2, \\
&= -1, \text{ as before.}
\end{aligned}$$

$$\begin{aligned}
\tan 180^\circ &= \tan (90^\circ + 90^\circ), \\
&= \frac{\sin}{\cos} (90^\circ + 90^\circ), \\
&= \frac{\sin 90^\circ . \cos 90^\circ + \cos 90^\circ . \sin 90^\circ}{\cos 90^\circ . \cos 90^\circ - \sin 90^\circ . \sin 90^\circ}, \\
&= \frac{1 . 0 + 0 . 1}{0 . 0 - 1 . 1}, \\
&= 0, \text{ as before.}
\end{aligned}$$

CHAPTER VIII. (Page 74.)

(1.) Express 14·741 grades in degrees, minutes, &c.

A grade is $\frac{1}{100}$ of a right angle,

a degree is $\frac{1}{90}$ of a right angle;

$\therefore$ 1 grade $= \frac{9}{10}$ of 1 degree;

$\therefore$ m grades $= \frac{9}{10}$ of m degrees;

$$\therefore 14{\cdot}741 \text{ grades} = \frac{9}{10} \text{ of } 14{\cdot}741 \text{ degrees},$$
$$= \frac{9 \times 14{\cdot}741}{10} \text{ degrees},$$
$$= 1{\cdot}4741 \times 9 \text{ degrees},$$
$$= 13{\cdot}2669 \text{ degrees}.$$

This may also be written as $13^\circ\ 16'\ 0{\cdot}84''$.

(2.) Express $14^\circ\ 14'\ 14''$ in grades, minutes, &c.

First $14^\circ 14' 14'' = 14{\cdot}2372^\circ$.

Then a degree $= \frac{1}{90}$ of a right angle,

a grade $= \frac{1}{100}$ of a right angle;

$\therefore$ 1 degree $= \frac{10}{9}$ of 1 grade;

$\therefore$ m degrees $= \frac{10}{9}$ of m grades;

$$\therefore 14\cdot2372^\circ = \frac{10}{9} \text{ of } 14\cdot2372 \text{ grades,}$$
$$= \frac{142\cdot372}{9} \text{ grades.}$$
$$= 15\cdot819 \text{ grades,}$$
$$= 15^g\ 49'\ 8\cdot4''.$$

(3.) In a right-angled triangle, the perpendicular equals half the hypothenuse. Express the angles in degrees and grades.

$\therefore$ hypothenuse = perpendicular × 2;

$\therefore$ the angles are respectively 90°, 60°, and 30°,

$\therefore$ they are also $\frac{10}{9}$ of 90^g, 60^g, and 30^g;

$\therefore$ they are 100^g, $66\cdot6^g$, and $33\cdot3^g$.

(4.) Find the ratio between 67° 14′ and 77^g 14′;
67° 14′ = 67 · 233°, and 77^g 14′ = 77 · 14 grades.

$$\text{A degree} = \frac{10}{9} \text{ of a grade,}$$
$$\therefore m \text{ degrees} = \frac{10}{9} \text{ of } m \text{ grades,}$$
$$\therefore 67\cdot233^\circ = \frac{10}{9} \text{ of } 77\cdot14 \text{ grades,}$$
$$= \frac{771.4}{9} \text{ grades,}$$
$$\therefore 67\cdot233^\circ = 85\cdot7^g,$$

and therefore the ratio of 67° 14′ to $77\cdot14^g$,

$$= \frac{85\cdot7}{77\cdot14};$$

i. e., they are as 85·7 to 77·14.

(5.) Express 101 grades in degrees, minutes, &c.

$$\text{A grade} = \frac{9}{10} \text{ of a degree;}$$

$$\therefore m \text{ grades} = \frac{9}{10} \text{ of } m \text{ degrees;}$$

$$\therefore 101 \text{ grades} = \frac{9}{10} \text{ of } 101 \text{ degrees.}$$

$$= 10{\cdot}1 \times 9 \text{ grades,}$$

$$= 90{\cdot}9 \text{ grades,}$$

$$= 90^{g}\ 54'.$$

(6.) Express 84° in grades, minutes, &c.

$$\text{A degree} = \frac{10}{9} \text{ of a grade;}$$

$$\therefore m \text{ degrees} = \frac{10}{9} \text{ of } m \text{ grades;}$$

$$\therefore 84^{\circ} \text{ degrees} = \frac{10}{9} \text{ of } 84 \text{ grades,}$$

$$= \frac{840}{9} \text{ grades,}$$

$$= 93{\cdot}33 \text{ grades,}$$

$$= 93^{g}\ 19'\ 48''.$$

CHAPTER IX. (Page 77.)

(1.) Express 36° 10′ as a fraction of ω°.

$$\omega = 57{\cdot}295 +{}^{\circ};$$

$$\therefore 30^{\circ}\ 10' = \frac{36{\cdot}16}{57{\cdot}29} \text{ of } \omega,$$

$$= {\cdot}631 \text{ of } \omega.$$

(2.) In a right-angled triangle, one acute angle is $\frac{3}{4}$ of the other. Find the three angles and express them in English, French, and circular methods.

Let A be the right angle, then—

$$B : C :: 3 : 4;$$

$$\therefore B = \frac{3}{7} \text{ of } 90^\circ,$$

and $$C = \frac{4}{7} \text{ of } 90^\circ.$$

Therefore, $$\begin{aligned} A &= 90^\circ, \\ B &= 38^\circ\, 34'\, 17'', \\ C &= 51^\circ\, 25'\, 43''; \end{aligned}$$

which is the expression in English measurement.

Also, $$B = \frac{3}{7} \text{ of } 100^g,$$

and $$C = \frac{4}{7} \text{ of } 100^g;$$

Therefore, $$\begin{aligned} A &= 100^g, \\ B &= 42^g\, 51'\, 25'', \\ C &= 57^g\ \ 8'\, 34''; \end{aligned}$$

which is the expression in the French method of measurement.

Lastly, $$90^\circ = \frac{\omega^\circ \times \pi}{2} = \frac{\pi}{2}\omega^\circ$$

$$\therefore B = \frac{3}{7} \text{ of } \frac{\omega^\circ \times \pi}{2} = \frac{3\pi}{14}\omega^\circ,$$

and $$C = \frac{4}{7} \text{ of } \frac{\omega^\circ \times \pi}{2} = \frac{2\pi}{7}\omega^\circ.$$

But, $\frac{\omega^\circ \times \pi}{2} = 90^\circ$.

$$\therefore A = 90^\circ,$$
$$B = 38^\circ\, 41'\, 17'',$$
$$C = 51^\circ\, 25'\, 43'',$$

which is the expression in terms of circular measurement.

(3.) Express $\frac{\omega^\circ}{4}$ in grades, also in degrees.

$$\frac{\omega^\circ}{4} = \frac{57\cdot295}{4},$$
$$= 14\cdot324.^\circ$$

But also $14\cdot324^\circ = \frac{10}{9}$ of $14\cdot324^g$;

$$\therefore\ 14\cdot324^\circ = 15\cdot704^g.$$

Therefore, $\frac{\omega^\circ}{4} = 14\cdot324^g = 15\cdot704^g$.

(4.) A circle has a radius of 8·47. Find the ratio of an arc of 4·27 to the circumference.

The circumference of a circle has to its diameter the ratio of 3·14159; *i. e.*—

$$\text{circumference} : \text{radius} :: 6\cdot28318 : 1;$$
$$\therefore \text{circumference} = \text{radius} \times 6\cdot28318;$$
$$\therefore \text{if the radius} = 8\cdot47,$$
$$\text{circumference} = 8\cdot47 \times 6\cdot28318,$$
$$= 53\cdot219,$$

and an arc of 4·27 is to the circumference as—

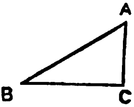
$$4\cdot27 : 53\cdot219;$$

$$\therefore \text{ the ratio} = \frac{4\cdot27}{53\cdot219},$$

$$\text{which is greater than } \frac{1}{13} \text{ but less than } \frac{1}{12}.$$

CHAPTER X. (Page 81.)

(1.) The *sine* of an angle = ·3784 of the radius. Find the angle.



$$\frac{\text{perpendicular}}{\text{hypothenuse}} = \frac{3784}{10000}.$$

Referring to the table on page 78, we find the *sine* of 23° 34′ to be ·3998158. Therefore the angle whose *sine* is ·3784 must be somewhat less than 23° 34′, and referring to a table of *natural sines,* we find—

$$nat.\ sin\ 22^\circ\ 14' = \cdot3783794,$$
$$nat.\ sin\ 22^\circ\ 15' = \cdot3786486;$$

therefore the given angle is greater than 22° 14′, and less than 22° 15′.

(2.) In a right-angled triangle the perpendicular is $\frac{1}{5}$ of the hypothenuse. Find the three angles.

Now, $$\frac{1}{5} = \cdot2,$$

and referring to the table on page 78,

we find *sine* of 11° 32′ = ·199938,
and *sine* of 11° 33′ = ·200223;

∴ ·2 is the *sine* of an angle between 11° 32′ and 11° 33′; and ∴ the other acute angle is between 78° 27′ and 78° 28″.

Therefore the three angles are respectively—
90°, 11° 32′ and 78° 28″.

(3.) The base of a right angle is $\frac{1}{3}$ of the hypothenuse, find the three angles.

Now $\frac{1}{3} = \cdot 33\dot{3}$, and referring to the table on page 78, we find—

sine of 17° 28′ = ·3001509;

∴ one of the given angles is greater than 17° 28′.

Now referring to a table of natural *sines*, we find—
sine of 19° 32′ = ·3332584,
sine of 19° 33′ = ·3335326;

∴ the given angle between 19° 32′ and 19° 33′.

Therefore, the three angles are, respectively,—
90°, 19° 32′ and 70° 28′.

(4.) In a right-angled triangle, B = 23° 34′, $a = 49 \cdot 4$, and A = 90. Solve the triangle.

First, C = 180° − A + B,
= 66° 26′.

Then the natural *sine* of 23° 34′ = ·3998158 of the radius (see table on page 79);

$$\begin{aligned}\therefore b &= \cdot 3998158 \text{ of } 49\cdot 4,\\ &= \frac{3998158}{10000000} \text{ of } 49\cdot 4,\\ &= \frac{4}{10} \text{ of } 49\cdot 4 \text{ (very nearly)},\\ &= 19\cdot 76.\end{aligned}$$

Also,

$$\begin{aligned}c &= \sqrt{a^2 - b^2},\\ &= \sqrt{49\cdot 4^2 \times 19\cdot 76^2},\\ &= \sqrt{2440 - 390\cdot 457},\\ &= \sqrt{2049\cdot 54},\\ &= 45\cdot 2.\end{aligned}$$

Therefore,

$$\begin{aligned}A &= 90^\circ,\\ B &= 23^\circ\, 34',\\ C &= 66^\circ\, 26',\\ a &= 49\cdot 4,\\ b &= 19\cdot 76,\\ c &= 45\cdot 2.\end{aligned}$$

CHAPTER XI. (Page 87.)

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

(1.) Given $C = 14^\circ\, 24'$, $a = 47\cdot 8$, $b = 39\cdot 4$. Required A, B, and c.

Now,

$$\frac{\sin A}{\sin B} = \frac{a}{b};$$

$$\therefore \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a - b}{a + b};$$

$$\therefore \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{a-b}{a+b}.$$

But—

$$\tan \tfrac{1}{2}(A+B) = \tan \tfrac{1}{2}(180^\circ - c),$$

$$= \cot \frac{c}{2};$$

$$\therefore \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2};$$

$$\therefore \log \tan \frac{A-B}{2} = \log (a-b) - \log (a+b) + \log \cot \frac{C}{2};$$

$$\therefore \text{L} \tan \frac{A-B}{2} = \log (a-b) - \log (a+b) + \text{L} \cot \frac{C}{2};$$

$$= \log 8{\cdot}4 - \log 87{\cdot}2 + \text{L} \cot 7^\circ\, 12',$$

$$= \text{L} \tan 37^\circ\, 19'\, 36'';$$

$$\therefore \frac{A-B}{2} = 37^\circ\, 19'\, 36'';$$

$$\therefore A - B = 74^\circ\, 39'\, 12'',$$

and $A + B = 165^\circ\, 36';$

$$\therefore 2A = 240^\circ\, 15'\, 12'',$$

and $A = 120^\circ\, 7'\, 36'',$

also $B = 45^\circ\, 28'\, 24''.$

Then, $\dfrac{c}{a} = \dfrac{\sin C}{\sin A};$

$$\therefore c = \frac{a \,.\, \sin C}{\sin A};$$

$$\therefore \log c = \log a + \text{L} \sin C - \text{L} \sin A,$$

$$= \log 47{\cdot}8 + \text{L} \sin 14^\circ\, 24' - \text{L}\, 120^\circ\, 7'\, 36'',$$

$$= \log 47{\cdot}8 + \text{L} \sin 14^\circ\, 24' - \text{L}\, 30^\circ\, 7'\, 36'',$$

$$= \log 23{\cdot}6841.$$

$$\therefore c = 23{\cdot}6841.$$

$$\text{Therefore,}\quad \begin{aligned} A &= 120^\circ\ \ 7'\ 36'', \\ B &= 45^\circ\ 28'\ 24'', \\ C &= 14^\circ\ 24' \\ a &= 47 \cdot 8, \\ b &= 39 \cdot 4, \\ c &= 23 \cdot 6841, \end{aligned}$$

and the triangle is completely known.

(2.) Given $B = 17^\circ\ 34'\ 18''$, $a = 23 \cdot 94$, $c = 17 \cdot 14$. Required A, C, and b.

$$\text{Now,}\quad \frac{\sin A}{\sin C} = \frac{a}{c};$$

$$\therefore \frac{\sin A - \sin C}{\sin A + \sin C} = \frac{a-c}{a+c},$$

and as before—

$$\tan\frac{A-C}{2} = \frac{a-c}{a+c}\cot\frac{B}{2};$$

$$\begin{aligned} \therefore L\tan\frac{A-C}{2} &= \log(a-c) - \log(a+c) + L\cot\frac{B}{2}, \\ &= \log 6 \cdot 8 - 41 \cdot 08 + L\cot 8^\circ\ 47'\ 9'', \\ &= L\tan 46^\circ\ 57'\ 50''; \end{aligned}$$

$$\therefore \frac{A-C}{2} = 46^\circ\ 57'\ 50'',$$

$$\begin{aligned} &\text{and} & A - C &= 93^\circ\ 55'\ 40'', \\ &\text{also} & A + C &= 162^\circ\ 25'\ 42'' = 180^\circ - B; \\ & & \therefore A &= 128^\circ\ 10'\ 41'', \\ &\text{and} & C &= 34^\circ\ 15'\ \ 1''. \end{aligned}$$

$$\text{Also,}\quad \frac{b}{c} = \frac{\sin B}{\sin C};$$

$$\therefore b = \frac{c \cdot \sin \mathrm{B}}{\sin \mathrm{C}};$$

$$\therefore \log b = \log c + \mathrm{L} \sin \mathrm{B} - \mathrm{L} \sin \mathrm{C},$$

$$= \log 17{\cdot}14 + \mathrm{L} \sin 17^\circ 34' 18'' - \mathrm{L} \sin 34^\circ 15' 1'',$$

$$= \log 11{\cdot}5704;$$

$$\therefore b = 11{\cdot}5704.$$

Therefore $\mathrm{A} = 128^\circ 10' 41''$
$\mathrm{B} = 17^\circ 34' 18''$
$\mathrm{C} = 34^\circ 15' 1''$
$a = 23{\cdot}94,$
$b = 11{\cdot}5704,$
$c = 17{\cdot}14,$
and the triangle is completely known.

(3.) Given $\mathrm{A} = 39^\circ 14' 29''$, $b = 87{\cdot}4$, $c = 84{\cdot}3$. Required B, C, and a.

As before—

$$\frac{\sin \mathrm{B}}{\sin \mathrm{C}} = \frac{b}{c};$$

$$\therefore \frac{\sin \mathrm{B} - \sin \mathrm{C}}{\sin \mathrm{B} + \sin \mathrm{C}} = \frac{b-c}{b+c};$$

$$\therefore \frac{\tan \frac{1}{2}(\mathrm{B}-\mathrm{C})}{\tan \frac{1}{2}(\mathrm{B}+\mathrm{C})} = \frac{b-c}{b+c};$$

and $\tan \frac{1}{2}(\mathrm{B}-\mathrm{C}) = \frac{b-c}{b+c} \cot \frac{\mathrm{A}}{2}$

$$\therefore \mathrm{L} \tan \frac{(\mathrm{B}-\mathrm{C})}{2} = \log(b-c) - \log(b+c) + \mathrm{L} \cot \frac{\mathrm{A}}{2},$$

$$= \log 3{\cdot}1 - \log 171{\cdot}7 + \mathrm{L} \cot 19^\circ 37' 14'',$$

$$= \mathrm{L} \tan 2^\circ 54' 2'';$$

$$\therefore \frac{B-C}{2} = 2^\circ\, 53'\, 59'',$$

and $B - C = 5^\circ\, 47'\, 58''$,

also $B + C = 140^\circ\, 45'\, 31'' = 180^\circ - A$;

$\therefore B = 73^\circ\, 16'\, 44''$,

and $C = 67^\circ\, 28'\, 46''$.

Again, $a = \dfrac{c\,.\,sin\ A}{sin\ c}$;

$$\therefore \log a = \log c + L\ sin\ A - L\ sin\ C,$$

$$= \log 84{\cdot}3 + L\ sin\ 39^\circ\, 14'\, 29'' - L\ sin\ 67^\circ\, 28'\, 46'',$$

$$= \log 57{\cdot}7196.$$

Therefore, $A = 39^\circ\, 14'\, 29''$,

$B = 73^\circ\, 16'\, 44''$,

$C = 67^\circ\, 28'\, 46''$,

$a = 57{\cdot}7196$,

$b = 87{\cdot}4$,

$c = 84{\cdot}3$.

(4.) Given $A = 14^\circ\, 3'\, 41''$, $b = 84{\cdot}41$, $a = 32{\cdot}7$.

Now, $\dfrac{sin\ B}{sin\ A} = \dfrac{b}{c}$;

$$\therefore sin\ B = \frac{b\,.\,sin\ A}{a};$$

$$\therefore L\ sin\ B = \log b + L\ sin\ A - \log a,$$

$$= \log 84{\cdot}41 + L\ sin\ 14^\circ\, 3'\, 4'' - \log 32{\cdot}7,$$

$$= L\ sin\ 38^\circ\, 48'\, 32'';$$

$\therefore B = 38^\circ\, 48'\, 32''$,

and $C = 180^\circ - (A + B)$,

$= 180^\circ - 52^\circ\, 52'\, 13''$,

$= 127^\circ\, 7'\, 47''$.

Again, $c = \dfrac{b \, . \, sin \text{ C}}{sin \text{ B}}$;

$\therefore \log c = \log b + \text{L } sin \text{ C} - \text{L } sin \text{ B}$,

$= \log 84{\cdot}41 + \text{L } sin \, 127° \, 7' \, 47'' - \text{L } sin \, 38° \, 48' \, 32''$

$= \log 81{\cdot}2983$,

$\therefore c = 81{\cdot}2983$, or $81{\cdot}2983 + 46{\cdot}2$.

Because B, from its *sine*, may be greater or less than 90°.

$$
\begin{aligned}
\text{Therefore} \quad \text{A} &= 14° \, 3' \, 41'', \\
\text{B} &= 38° \, 48' \, 32'', \\
\text{C} &= 127° \, 7' \, 47'', \\
a &= 32{\cdot}7, \\
b &= 84{\cdot}41, \\
c &= 81{\cdot}2983 + 46{\cdot}2 = 127{\cdot}4983.
\end{aligned}
$$

(5.) Given $\text{B} = 37° \, 8' \, 32''$, $a = 39{\cdot}47$, $b = 84{\cdot}32$. Required A, C, and c.

Then, $\dfrac{sin \text{ A}}{sin \text{ B}} = \dfrac{a}{b}$;

$$
\begin{aligned}
\therefore sin \text{ A} &= \frac{a \, . \, sin \text{ B}}{b}, \\
&= \frac{39{\cdot}47}{84{\cdot}32} \, sin \text{ B} \\
&= \frac{39{\cdot}47}{84{\cdot}32} \, sin \, 37° \, 8' \, 32''; \\
\therefore \text{L } sin \text{ A} &= \log a + \text{L } sin \text{ B} - sin \, b, \\
&= \log 39{\cdot}47 + \text{L } sin \, 37° \, 8' \, 32'' - sin \, 84{\cdot}32, \\
&= \text{L } 16° \, 25' \, 3''; \\
\therefore \text{A} &= 16° \, 25' \, 3''.
\end{aligned}
$$

$$
\begin{aligned}
\text{Then,} \quad \text{C} &= 180° - (\text{A} + \text{B}), \\
&= 126° \, 26' \, 25'',
\end{aligned}
$$

and $$c = \frac{b\,.\,\sin \mathrm{C}}{\sin \mathrm{B}};$$

$$\therefore \log c = \log b + \mathrm{L}\sin \mathrm{C} - \mathrm{L}\sin \mathrm{B},$$
$$= \log 84{\cdot}32 + \mathrm{L}\sin 36^\circ\,26'\,25'' - \mathrm{L}\sin 37^\circ\,8'\,32'',$$
$$= \log 82{\cdot}9499;$$
$$\therefore c = 82{\cdot}9499.$$

Therefore $\mathrm{A} = 16^\circ\,25'\,3''$,
$\mathrm{B} = 37^\circ\,8'\,32''$,
$\mathrm{C} = 126^\circ\,26'\,25''$,
$a = 39{\cdot}47$,
$b = 84{\cdot}32$,
$c = 82{\cdot}9499$.

(6.) Given $\mathrm{C} = 41^\circ\,13'\,42''$, $b = 14{\cdot}27$, $c = 39{\cdot}41$. Required A, B, and a.

First, $$\frac{\sin \mathrm{B}}{\sin \mathrm{C}} = \frac{b}{c};$$

$$\therefore \sin \mathrm{B} = \frac{b \sin \mathrm{C}}{c};$$

$$\therefore \mathrm{L}\sin \mathrm{B} = \log b + \mathrm{L}\sin \mathrm{C} - \log c,$$
$$= \log 14{\cdot}27 + \mathrm{L}\sin 41^\circ\,13'\,42'' - \log 39{\cdot}41,$$
$$= \mathrm{L}\sin 13^\circ\,48'\,10'',$$
$$\therefore \mathrm{B} = 13^\circ\,48'\,10''.$$

Then $$\mathrm{A} = 180^\circ - (\mathrm{A} + \mathrm{B}),$$
$$= 124^\circ\,58'\,8''.$$

and $$\frac{a}{c} = \frac{\sin \mathrm{A}}{\sin \mathrm{C}};$$

$$\therefore a = \frac{c\,.\,\sin \mathrm{A}}{\sin \mathrm{C}};$$

$$\therefore \log a = \log c + \mathrm{L}\,sin\,\mathrm{A} - \mathrm{L}\,sin\,\mathrm{C},$$
$$= \log 39{\cdot}41 + \mathrm{L}\,sin\,34^\circ\,58'\,8'' - \mathrm{L}\,sin\,41^\circ\,13'\,42'';$$
$$\therefore a = 34{\cdot}2811.$$

Therefore, $\mathrm{A} = 124^\circ\,58'\,\ 8''$,
$\mathrm{B} = 13^\circ\,48'\,10''$,
$\mathrm{C} = 41^\circ\,13'\,42''$,
$a = 34{\cdot}2811$,
$b = 14{\cdot}27$,
$c = 39{\cdot}41$.

(7.) Given $\mathrm{A} = 14^\circ\,31'\,14''$, $\mathrm{B} = 80^\circ\,1'\,4''$, $c = 39{\cdot}78$.

First $\mathrm{C} = 180^\circ - (\mathrm{A} + \mathrm{B})$,
$= 85^\circ\,27'\,42''$.

Then, $$\frac{b}{c} = \frac{sin\,\mathrm{B}}{sin\,\mathrm{C}};$$
$$\therefore b = \frac{c\,.\,sin\,\mathrm{B}}{sin\,\mathrm{C}};$$
$$\therefore \log b = \log c + \mathrm{L}\,sin\,\mathrm{B} - \mathrm{L}\,sin\,\mathrm{C},$$
$$= \log 39{\cdot}78 + \mathrm{L}\,sin\,80^\circ\,1'\,4'' - \mathrm{L}\,sin\,85^\circ\,27'\,42'',$$
$$= \log 39{\cdot}301;$$
$$= 39{\cdot}301.$$

Also, $$\frac{a}{b} = \frac{sin\,\mathrm{A}}{sin\,\mathrm{B}};$$
$$\therefore a = \frac{b\,.\,sin\,\mathrm{A}}{sin\,\mathrm{B}};$$
$$\therefore \log a = \log b + \mathrm{L}\,sin\,\mathrm{A} - \mathrm{L}\,sin\,\mathrm{B},$$

$$= \log 39\cdot301 + \text{L}\ sin\ 14° 31' 14'' - \text{L}\ sin\ 80° 1' 4'',$$
$$= \log 10\cdot0009;$$
$$\therefore a = 10\cdot0009.$$

Therefore,
$$\text{A} = 14° 31' 14'',$$
$$\text{B} = 80° \ 1' \ 4'',$$
$$\text{C} = 85° 27' 42'',$$
$$a = 10\cdot0009,$$
$$b = 39\cdot301,$$
$$c = 39\cdot78.$$

(8.) Given $\text{A} = 89° 14' 14''$, $\text{C} = 30° 40'$, $b = 41\cdot69$. Required, B, a, and c.

First,
$$\text{B} = 180° - (\text{A} + \text{C}),$$
$$= 60° 5' \ 46''.$$

Then,
$$\frac{a}{b} = \frac{sin\ \text{A}}{sin\ \text{B}};$$
$$\therefore \quad a = \frac{b\, .\, sin\ \text{A}}{sin\ \text{B}};$$
$$\therefore \log a = \log b + \text{L}\ sin\ \text{A} - \text{L}\ sin\ \text{B},$$
$$= \log 41\cdot69 + \text{L}\ sin\ 89° 14' 14'' - \text{L}\ sin\ 60° 5' 46'',$$
$$= \log 48\cdot0887,$$
$$\therefore a = 48\cdot0887.$$

Also,
$$\frac{c}{b} = \frac{sin\ \text{C}}{sin\ \text{A}};$$
$$\therefore c = \frac{a\, .\, sin\ \text{C}}{sin\ \text{A}}.$$

$$\therefore \log c = \log a + L\, sin\, C - L\, sin\, A,$$

$$= \log 48{\cdot}0887 + L\, sin\, 30^\circ 40' - L\, sin\, 89^\circ 14' 14''$$

$$= \log 24{\cdot}5407;$$

$$\therefore c = 24{\cdot}5407.$$

Therefore,

$$A = 89^\circ\ 14'\ 14'',$$

$$B = 60^\circ\ \ 5'\ 46'',$$

$$C = 30^\circ\ 40',$$

$$a = 48{\cdot}0887,$$

$$b = 41{\cdot}69,$$

$$c = 24{\cdot}5407.$$

PART II.

(1.) Prove $\frac{sin}{cos} = \frac{1}{\sqrt{cosec^2 - 1}}$,

$$sin = \frac{1}{cosec},$$

$$cos = \sqrt{1 - sin^2} = \sqrt{1 - \frac{1}{cosec^2}},$$

$$= \frac{\sqrt{cosec^2 - 1}}{cosec};$$

$$\therefore \frac{sin}{cos} = \frac{1}{cosec} \div \frac{\sqrt{cosec^2 - 1}}{cosec},$$

$$= \frac{1}{\sqrt{cosec^2 - 1}}.$$

(2.) Prove $cot^2 . cos^2 = cot^2 - cos^2$,

$$cot = \frac{\sqrt{1 - sin^2}}{sin}; \quad \therefore cot^2 = \frac{1 - sin^2}{sin^2},$$

$$cos = \sqrt{1 - sin^2}; \quad \therefore cos^2 = 1 - sin^2;$$

$$\therefore cot^2 \times cos^2 = \frac{1 - sin^2}{sin^2} \times (1 - sin^2),$$

and $cot^2 - cos^2 = \frac{1 - sin^2}{sin^2} - (1 - sin^2)$.

But $\frac{1 - sin^2}{sin^2} \times (1 - sin^2) = \frac{1 - 2\, sin^2 + sin^4}{sin^2}$;

also, $\frac{1 - sin^2}{sin^2} - (1 - sin^2) = \frac{1 - 2\, sin^2 + sin^4}{sin^2}$,

$$\therefore cot^2 \times cos^2 = cot^2 - cos^2.$$

(3.) Prove $sec^2 \times cosec^2 = sec^2 + cosec^2$,

$$sec = \frac{1}{\sqrt{1 - sin^2}}; \quad \therefore sec^2 = \frac{1}{1 - sin^2},$$

$$cosec = \frac{1}{sin}; \quad \therefore cosec^2 = \frac{1}{sin^2},$$

Then, $sec^2 \times cosec^2 = \frac{1}{1 - sin^2} \times \frac{1}{sin^2}$,

$$= \frac{1}{sin^2 - sin^4},$$

and $sec^2 + cosec^2 = \frac{1}{1 - sin^2} + \frac{1}{sin^2}$,

$$= \frac{sin^2 + 1 - sin^2}{sin^2 - sin^4},$$

$$= \frac{1}{sin^2 - sin^4};$$

$$\therefore sec^2 \times cosec^2 = sec^2 + cosec^2.$$

(4.) Prove $tan\ 60^\circ = \frac{tan\ 20^\circ + tan\ 40^\circ}{1 - tan\ 20^\circ\ tan\ 40^\circ}$.

For let $20^\circ = A$, $40^\circ = B$, and $60^\circ = A + B$.

Then $tan\ (A + B) = \frac{sin\ (A + B)}{cos\ (A + B)}$,

$$= \frac{sin\ A\ cos\ B + cos\ A\ sin\ B}{cos\ A\ cos\ B - sin\ A\ sin\ B},$$

(and dividing numerator and divisor by cos A cos B)

$$= \frac{\frac{sin\ A\ cos\ B}{cos\ A\ cos\ B} + \frac{cos\ A\ sin\ B}{sin\ A . sin\ B}}{\frac{cos\ A\ cos\ B}{cos\ A\ cos\ B} - \frac{sin\ A\ sin\ B'}{cos\ A\ cos\ B}}$$

$$= \frac{tan\ A + tan\ B}{1 - tan\ A\ tan\ B}.$$

In same way $tan\ A - B = \dfrac{tan\ A - tan\ B}{1 + tan\ A\ tan\ B}$;

$\therefore\ tan\ 60^\circ = \dfrac{tan\ 20^\circ + tan\ 40^\circ}{1 - tan\ 20^\circ\ tan\ 40^\circ}$,

also, $tan\ 20^\circ\ \dfrac{tan\ 40^\circ - tan\ 20^\circ}{1 + tan\ 40^\circ\ tan\ 20^\circ}$.

(5.) Prove $(tan\ 45^\circ + A) = \dfrac{1 + tan\ A}{1 - tan\ A}$.

For, $tan\ (A + B) = \dfrac{sin\ (A + B)}{cos\ (A + B)}$,

$$= \frac{tan\ A + tan\ B}{1 - tan\ A\ tan\ B},$$

$$\therefore\ tan\ (45^\circ + A) = \frac{tan\ 45^\circ + tan\ A}{1 - tan\ 45^\circ\ tan\ A};$$

$$\therefore\ tan\ 45^\circ = 1, = \frac{1 + tan\ A}{1 - tan\ A}.$$

In same way $tan\ 45^\circ - A = \dfrac{1 - tan\ A}{1 + tan\ A}$.

(6.) Prove $\frac{1 + \sin A}{\cos A} = \tan\left(45^\circ + \frac{A}{2}\right)$,

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2},$$

and $\cos^2 + \sin^2$ may always be written for 1;

$$\therefore \frac{1 + \sin A}{\cos A} = \frac{\cos^2 + \sin^2 + 2 \sin \frac{A}{2} \cos \frac{A}{2}}{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}},$$

$$= \frac{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2}}{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}.$$

But, $\cos^2 + \sin^2 + 2 \sin \cos = (\cos + \sin)^2$,

$$= (\cos + \sin)(\cos + \sin);$$

also, $\cos^2 - \sin^2 = (\cos + \sin)(\cos - \sin)$;

$$\therefore \frac{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}\, 2 \sin \frac{A}{2} \cos \frac{A}{2}}{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}} = \frac{(\cos + \sin)(\cos + \sin)}{(\cos + \sin)(\cos - \sin)},$$

and dividing by $(\cos + \sin) = \frac{\cos \frac{A}{2} + \sin \frac{A}{2}}{\cos \frac{A}{2} - \sin \frac{A}{2}}$,

and dividing both by $\cos \frac{A}{2} = \frac{1 + \tan \frac{A}{2}}{1 - \tan \frac{A}{2}}$.

But, $$tan\ 45^\circ + \frac{A}{2} = \frac{1 + tan\ \frac{A}{2}}{1 - tan\ \frac{A}{2}};$$

$$\therefore \frac{1 + sin\ A}{cos\ A} = tan\left(45^\circ + \frac{A}{2}\right).$$

(7.) Given $a = 14{\cdot}713$, $c = 7{\cdot}81$, $B = 47^\circ$.
Required A, C, and b.

Then, $$\frac{sin\ A}{sin\ C} = \frac{a}{c};$$

$$\therefore \frac{sin\ A - sin\ C}{sin\ A + sin\ C} = \frac{a-c}{a+c};$$

$$\therefore \frac{tan\ \frac{1}{2}(A - C)}{tan\ \frac{1}{2}(A + C)} = \frac{a-c}{a+c}.$$

But $$tan\ \tfrac{1}{2}(A + C) = tan\ \tfrac{1}{2}(180 - B),$$
$$= cot\ \frac{B}{2};$$

$$\therefore tan\ \tfrac{1}{2}(A - C) = \frac{a-c}{a+c} \cdot cot\ \frac{B}{2},$$
$$= \frac{6{\cdot}903}{22{\cdot}523} \cdot cot\ 23^\circ\ 30';$$

$$\therefore L\,tan\ \tfrac{1}{2}(A - C) = \log 6{\cdot}903 - \log 22{\cdot}523 + L\,cot\ 23^\circ 30',$$
$$= 9{\cdot}0871114,$$
$$= L\ tan\ 6^\circ\ 58'\ 3'';$$

$$\therefore \tfrac{1}{2}(A - C) = 6^\circ\ 58'\ 3'';$$

$$\therefore A - C = 13^\circ\ 56'\ 6'',$$

and $$A + C = 180^\circ - B,$$
$$= 133^\circ;$$

$$\therefore A = 73^\circ\ 28'\ 3'',$$

and $$C = 59^\circ\ 31'\ 57''.$$

Then, $\frac{b}{c} = \frac{\sin B}{\sin C}$;

$\therefore b = \frac{c \cdot \sin B}{\sin C}$,

$= \frac{7 \cdot 81 \cdot \sin 47^\circ}{\sin 59^\circ 31' 57''}$;

$\therefore \log b = \log 7 \cdot 81 + L \sin 47^\circ - L \sin 59^\circ 31' 57''$;

$\therefore b = 6 \cdot 625$.

Then, $A = 73^\circ 28' 3''$,
$B = 47^\circ$,
$C = 59^\circ 31' 57''$;
$a = 14 \cdot 713$,
$b = 6 \cdot 625$,
$c = 7 \cdot 81$.

(8.) Given b, c, and B, to find A, C, and a.

Let $b = 14 \cdot 71$, $c = 18 \cdot 714$, and $B = 39^\circ 4'$.

Then, $\frac{\sin C}{\sin B} = \frac{c}{b}$;

$\therefore \sin C = \frac{c}{b} \sin B$;

$\therefore L \sin C = \log 18 \cdot 714 + L \sin 39^\circ 4' - \log 14 \cdot 71$,

$= L \sin 53^\circ 20' 24''$, or $126^\circ 42' 26''$;

$\therefore C = 53^\circ 17' 34''$, or $126^\circ 42' 26''$.

Then, $A = 180^\circ - (B + C)$,
$= 180^\circ - 92^\circ 21' 34''$,
or $180^\circ - 165^\circ 46' 26''$,
$= 87^\circ 38' 26''$, or $14^\circ 13' 34''$.

Taking the smaller value, $a = \dfrac{b \,.\, \sin A}{\sin B}$;

$$\begin{aligned} \therefore \log a &= \log b + \mathrm{L} \sin A - \mathrm{L} \sin B, \\ &= \log 14{\cdot}71 + \mathrm{L} \sin 87^\circ 38' 26'' - \mathrm{L} \sin 39^\circ 4', \\ &= \log 23{\cdot}321; \\ \therefore a &= 23{\cdot}321. \end{aligned}$$

$$\begin{aligned} \text{Or, } \log a &= \log 14{\cdot}71 + \mathrm{L} \sin 14^\circ 13' 34'' - \mathrm{L} \sin 39^\circ 4', \\ &= \log 5{\cdot}736; \\ \therefore a &= 5{\cdot}736. \end{aligned}$$

Therefore,

$A = 87^\circ 38' 26''$,
$B = 39^\circ 4'$,
$C = 53^\circ 17' 34''$,
$a = 23{\cdot}321$,
$b = 14{\cdot}71$,
$c = 18{\cdot}714$.

Or,

$A = 14^\circ 13' 34''$,
$B = 39^\circ 4'$,
$C = 126^\circ 42' 26''$,
$a = 5{\cdot}736$,
$b = 14{\cdot}71$,
$c = 18{\cdot}714$.

(9.) $A = 49^\circ 1'$, $B = 37^\circ 5'$, $c = 37{\cdot}4$.

Find C, a, and b.

$$\begin{aligned} \text{Then, } C &= 180^\circ - (A + B), \\ &= 93^\circ 54'; \end{aligned}$$

also, $b = \frac{c \,.\, sin\ B}{sin\ C}$;

$\therefore \log b = \log c + L\,sin\ B - L\,sin\ C,$

$= \log 37{\cdot}4 + L\,sin\ 37^\circ\ 5' - L\,sin\ 93^\circ\ 54',$

$= \log 22{\cdot}65;$

---

and $a = \frac{b \,.\, sin\ A}{sin\ B}$;

$\therefore \log a = \log 22{\cdot}65 + L\,sin\ 49^\circ\ 1' - L\,sin\ 37^\circ\ 5',$

$= \log 28{\cdot}35;$

$\therefore a = \log 28{\cdot}35.$

Therefore,
$A = 49^\circ\ 1',$
$B = 37^\circ\ 5',$
$C = 93^\circ\ 54',$
$a = 28{\cdot}35,$
$b = 22{\cdot}65,$
$c = 37{\cdot}4.$

(10.) Given B, C, and b, to find A, a, and c.

Let $B = 95^\circ\ 3'$, $C = 14^\circ\ 32'$, and $b = 7{\cdot}8$.

Then $A = 180^\circ - (B + C),$

$= 70^\circ\ 25';$

also, $a = \frac{b \cdot sin\ A}{sin\ B}$;

$\therefore \log a = \log 7{\cdot}8 + L\,sin\ 70^\circ\ 25' - L\,sin\ 95^\circ\ 3',$

(But $sin\ 95^\circ\ 3' = 90^\circ - 5^\circ\ 3' = 84^\circ\ 57'$.)

$= \log 7{\cdot}377;$

$\therefore a = 7{\cdot}877,$

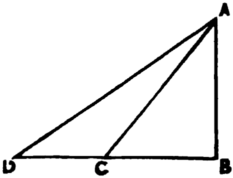
---

and $c = \frac{b \,.\, sin\ C}{sin\ B}$;

$$\therefore \log c = \log b + \text{L} \sin \text{C} - \text{L} \sin \text{B},$$
$$= \log 7{\cdot}8 + \text{L} \sin 14^\circ\, 32 - \text{L} \sin 84^\circ\, 57'$$
$$= \log 1{\cdot}964;$$
$$\therefore c = 1{\cdot}964.$$

Therefore,
$$\text{A} = 70^\circ\, 25',$$
$$\text{B} = 95^\circ\, 3',$$
$$\text{C} = 14^\circ\, 32',$$
$$a = 7{\cdot}377,$$
$$b = 7{\cdot}8,$$
$$c = 1{\cdot}964.$$

(11.) A castle is surrounded by a moat. Find the height of the castle and the width of the moat.



Let A B be the castle, and B C the moat. Let the angle A C B be measured, also the angle A D B; D C being any distance measured along the ground from C, in the line B C.

Let A C B = 60°, A D B = 40°, D C = 50 feet.

Then, A C D = 120°, and in the triangle A D C we have two angles and the included side.

Thus, $a = 50^\circ$, D = 40°, C = 120°.

Then, $\text{A} = 180^\circ - (120^\circ + 40^\circ) = 20^\circ$,

and
$$d = \frac{a \,.\, \sin \text{D}}{\sin \text{A}},$$
$$= \frac{50 \,.\, \sin 40}{\sin 20};$$

$$\therefore \log d = \log 50 + \text{L} \sin 40^\circ - \text{L} \sin 20^\circ,$$
$$= \log 93 \cdot 97,$$
$$\therefore d = 93 \cdot 97.$$

Then in triangle A C B we have the side A C, which was d in triangle A D C, but is b in triangle A C B, also the two angles C and B.

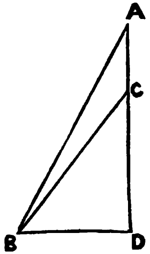
Thus, $b = 93 \cdot 97$, $C = 60$, $B = 90$.

Then, $c = \dfrac{b \cdot \sin \text{C}}{\sin \text{B}}$;

$$\therefore \log c = \log 93 \cdot 97 + \text{L} \sin 60^\circ - \text{L} \sin 90^\circ;$$
$$\therefore \log c = \log 81 \cdot 38;$$
$$\therefore c = 81 \cdot 38,$$ which is the height of the castle.

Then $a = \sqrt{b^2 - c^2}$,
$= \sqrt{2207 \cdot 65}$,
$= 46 \cdot 9$, which is the width of the moat.

(12.) A C is a flagstaff on the top of a tower C D. Required the height of each.



Let B D be a measured distance of 70 feet, the angle C B D = 49°, and the angle A B D = 57°.

Then in triangle A B D we have all the angles and one side given.

Thus, $a = 70$, $B = 57°$, $D = 90°$, and $A = 33°$.

$$\text{Then, } b = \frac{a \,.\, sin\ B}{sin\ A},$$

$$= \frac{70 \,.\, sin\ 57°}{sin\ 33°};$$

$$\therefore \log b = \log 70 + L\ sin\ 57° - L\ sin\ 33°,$$

$$= \log 107 {\cdot} 78;$$

$\therefore b = 107{\cdot}78$, which is the total of tower and flagstaff.

Then, in triangle C B D, we have—

$$c = 70°, B = 49°, \text{ and } D = 90°;$$

$$\therefore C = 41°,$$

$$\text{and} \qquad b = \frac{c \,.\, sin\ B}{sin\ C};$$

$$\therefore \log b = \log c + L\ sin\ B - L\ sin\ C,$$

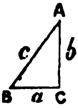
$$= \log 70 + L\ sin\ 49° - L\ sin\ 41°,$$

$= 80{\cdot}526$ feet, which is the height of the tower;

$$\therefore \text{flagstaff} = 107{\cdot}78 - 80{\cdot}526,$$

$= 27{\cdot}254$ feet.

(13.) Required the width of a marsh. Take any



point A on one side, and any points B, C, on the other. Measure B C.

Let $B = 35°$, $C = 71°$, and $a = 179$ yards.

Then, $A = 180° - 74°$;

---

also, $$b = \frac{a \,.\, sin\ B}{sin\ A};$$

$$\therefore \log b = \log 179 + L\ sin\ 35° - L\ sin\ 74°,$$
$$= \log 1068;$$
$$\therefore b = 1068 \text{ yards},$$

---

and $$c = \frac{a \,.\, sin\ C}{sin\ A};$$

$$\therefore \log c = \log 179 + L\ sin\ 71° - L\ sin\ 74°,$$
$$= 1760 \cdot 6.$$

Then in the two triangles A B D, A C D, we know, in each, one side and two angles, and can therefore find A D, which is the width of the marsh. Thus, in A B D, we have—

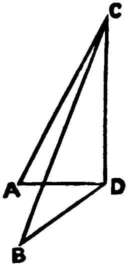
$B = 35°$, $D = 90°$, and A B, or $d = 1760 \cdot 6$.

Then, $$\frac{A\,D}{A\,B} = \frac{sin\ 35°}{sin\ 90°};$$

$$\therefore A\,D = A\,B \,.\, sin\ 35°;$$
$$\therefore \log A\,D = \log 1760{\cdot}6 + L\ sin\ 35° - 10.$$
$$= \log 100 \cdot 9;$$
$$\therefore A\,D = 100 \cdot 9.$$

Therefore the width of the marsh from A to D is 100·9 yards.

(14.) The spire of a church, 200 feet high, subtends at A, on the north, an angle of 24°, and at B,



on the east, an angle of 31°. Required the distance of A from B.

In triangle C D B we have—

$$CD = 200,\ B = 31^\circ,\ D = 90^\circ.$$

Then C = 59°, and—

$$BD = \frac{CD \,.\, \sin C}{\sin B};$$

$$\therefore \log BD = \log 200 + L \sin 59^\circ - L \sin 31^\circ,$$
$$= \log 332{\cdot}8;$$
$$\therefore BD = 332{\cdot}8.$$

In triangle C D A, we have—

$$CD = 200,\ A = 24^\circ,\ D = 90^\circ;$$
$$\therefore C = 66^\circ, \text{ and}$$
$$AD = \frac{CD \,.\, \sin C}{\sin A};$$
$$\therefore \log AD = \log 200 + L \sin 66^\circ - L \sin 24^\circ,$$
$$= \log 448{\cdot}48.$$

Then, having found B D and A D, we have, in the triangle A D B,

$$BD = 332{\cdot}8,\ AD = 448{\cdot}48, \text{ and } D = 90^\circ.$$

The third side, A B, will be the distance between the two points of observation.

Then, $$AB^2 = AD + BD^2$$

and $$AB = \sqrt{AD^2 + BD^2}$$
$$= 558{\cdot}4 \text{ yards},$$

which is the distance between A and B.

(15.) From the ratios of 90° and 60° find those of 150° and 30°,

$$150° = 90° + 60°,$$

and $$\sin 150° = \sin 90°, \cos 60° + \cos 90° \,.\, \sin 60°$$
$$= 1 \times \frac{1}{2} + 0 \times \frac{\sqrt{3}}{2},$$
$$= \frac{1}{2},$$

and this is evidently correct, for 150° = 180° − 30°, therefore the ratios of 150° are equal in magnitude to those of 30°.

Having found the *sin* from the ratios of 90° and 60°, we may find the other ratios in the ordinary way: thus—

$$\cos = \sqrt{1 - \sin^2},$$
$$= \sqrt{1 - \frac{1}{4}},$$
$$= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

$$\tan = \frac{\sin}{\cos} = \frac{1}{2} \div \frac{2}{\sqrt{3}},$$
$$= \frac{1}{\sqrt{3}}.$$

$$\cot = \frac{1}{\tan} = \sqrt{3}.$$

$$sec = \frac{1}{cos} = \frac{2}{\surd 3}.$$

$$cosec = \frac{1}{sin} = 2.$$

But though the magnitudes are equal, the signs are different. For 30°, all the ratios are + ; for 150°, all but the *sine* and *cosecant* are −. But the above formula should give the sign as well as the magnitude, as we find it does.

$$\text{For } cos = \pm\sqrt{\frac{3}{4}}.$$

$$= \frac{-\surd 3}{4}.$$

We choose the − sign because we know 150° to be in the second quadrant.

$$\text{Then,} \qquad tan = \frac{sin}{cos} = \frac{1}{2} \div \frac{\surd 3}{2},$$

$$= \frac{-1}{\surd 3},$$

and the remaining ratios follow from these with correct signs.

But, if the angle be 30°, we choose the + sign for the *cos*, and we find all the rest have the same sign.

(16.) From the ratios of 90° and 30°, find those of 120° and 60°.

$$120^\circ = 90^\circ + 30^\circ,$$

$$60^\circ = 90^\circ - 30^\circ;$$

$$\therefore sin\ 120^\circ = sin\ 90^\circ\ cos\ 30^\circ + cos\ 90^\circ\ sin\ 30^\circ,$$

$$= 1 \times \frac{\surd 3}{2} + 0 \times \frac{1}{2},$$

$$= \frac{\surd 3}{2},$$

and $$\sin 60^\circ = \sin 90^\circ \cos 30^\circ - \cos 90^\circ \sin 30^\circ,$$
$$= 1 \times \frac{\sqrt{3}}{2} + 0 \times \frac{1}{2},$$
$$= \frac{\sqrt{3}}{2},$$

so that sin 60° and sin 120° are the same, both in sign and magnitude.

Again, $$\cos 120^\circ = \cos 90^\circ \cos 30^\circ - \sin 90^\circ \sin 30^\circ,$$
$$= 0 \times \frac{\sqrt{3}}{2} - 1 \times \frac{1}{2};$$

and $$\cos 60^\circ = \cos 90^\circ \cos 30^\circ + \sin 90^\circ \sin 30^\circ,$$
$$= 0 \times \frac{\sqrt{3}}{2} + 1 \times \frac{1}{2};$$

$$\therefore \cos 120^\circ = -\frac{1}{2},$$

and $$\cos 60^\circ = +\frac{1}{2}.$$

From $$\sin 120^\circ = \frac{\sqrt{3}}{2},\ \cos 120^\circ = -\frac{1}{2},$$
$$\sin 60^\circ = \frac{\sqrt{3}}{2},\ \cos 60^\circ = \frac{1}{2},$$

we get, very easily, the remaining ratios of both 60° and 120°.

(17.) From the ratios of 60° and 30°, find those of 90°.

$$90^\circ = 60^\circ + 30^\circ;$$
$$\therefore \sin 90^\circ = \sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ,$$
$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2},$$
$$= \frac{3}{4} + \frac{1}{4},$$
$$= 1.$$

From $\sin 90° = 1$, we get—

$$\begin{aligned} \cos &= \sqrt{1 - \sin^2}, \\ &= \sqrt{0}, \\ &= 0. \end{aligned}$$

$$\tan = \frac{\sin}{\cos} = \frac{1}{0} = \infty.$$

$$\cot = \frac{1}{\tan} = \frac{1}{\infty} = 0.$$

$$\sec = \frac{1}{\cos} = \frac{1}{0} = \infty.$$

$$\operatorname{cosec} = \frac{1}{\sin} = \frac{1}{1} = 1.$$

(18.) From the ratios of 90° and 45°, find those of 135°.

$$135° = 90° + 45°;$$

$$\begin{aligned} \therefore \sin 135° &= \sin 90° \cos 45° + \cos 90° \sin 45°, \\ &= 1 \times \frac{1}{\sqrt{2}} + 0 \times \frac{1}{\sqrt{2}}, \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

---

and

$$\begin{aligned} \cos 135° &= \cos 90° \cos 45° - \sin 90° \sin 45°, \\ &= 0 \times \frac{1}{\sqrt{2}} - 1 \times \frac{1}{\sqrt{2}}, \\ &= \frac{-1}{\sqrt{2}}. \end{aligned}$$

We see from this, and previous examples, that *cos* as derived from $\sqrt{1 - sin^2}$ is uncertain as to sign, but as derived from $cos\ a\ cos\ b - sin\ a\ sin\ b$ is correct in sign as well as in magnitude.

From *sin* and *cos* of 135°, as above, we get—

$$
\begin{aligned}
tan &= \frac{sin}{cos} = 1,\\
cot &= \frac{cos}{sin} = 1,\\
sec &= \frac{1}{sec} = \sqrt{2},\\
cosec &= \frac{1}{sin} = \sqrt{2}.
\end{aligned}
$$

(19.) From the ratios of 60° and 45° find those of 105° and 15°.

$$
\begin{aligned}
105° &= 60° + 45°,\\
15° &= 60° - 45°.
\end{aligned}
$$

Then
$$
\begin{aligned}
sin\ 105° &= sin\ 60°\ cos\ 45° + cos\ 60°\ sin\ 45°,\\
&= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}},\\
&= \frac{\sqrt{3}}{\sqrt{8}} + \frac{1}{\sqrt{8}},\\
&= \frac{1 + \sqrt{3}}{\sqrt{8}}.
\end{aligned}
$$

$$
\begin{aligned}
cos\ 105° &= cos\ 60°\ cos\ 45° - sin\ 60°\ sin\ 45°,\\
&= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}},\\
&= \frac{1}{\sqrt{8}} - \frac{\sqrt{3}}{\sqrt{8}},\\
&= \frac{1 - \sqrt{3}}{\sqrt{8}} = \frac{-(\sqrt{3} - 1)}{\sqrt{8}}.
\end{aligned}
$$

Also,

$$\begin{aligned}
\sin 15^\circ &= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ,\\
&= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}},\\
&= \frac{\sqrt{3}}{\sqrt{8}} - \frac{1}{\sqrt{8}},\\
&= \frac{\sqrt{3}-1}{\sqrt{8}},
\end{aligned}$$

$$\begin{aligned}
\cos 15^\circ &= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ,\\
&= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}},\\
&= \frac{1}{\sqrt{8}} + \frac{\sqrt{3}}{\sqrt{8}},\\
&= \frac{1+\sqrt{3}}{\sqrt{8}}.
\end{aligned}$$

From these we get, very readily, the remaining ratios of 105° and 15°: thus—

$$\begin{aligned}
\tan 105^\circ &= \frac{\sin}{\cos},\\
&= \frac{1+\sqrt{3}}{\sqrt{8}} \div \frac{1-\sqrt{3}}{\sqrt{8}},\\
&= \frac{1+\sqrt{3}}{1-\sqrt{3}}.
\end{aligned}$$

$$\begin{aligned}
\tan 15^\circ &= \frac{\sin}{\cos},\\
&= \frac{\sqrt{3}-1}{\sqrt{8}} \div \frac{1+\sqrt{3}}{\sqrt{8}},\\
&= \frac{\sqrt{3}-1}{1+\sqrt{3}}.
\end{aligned}$$

Then, $sec = \frac{1}{cos}, \; cosec = \frac{1}{sin}, cot = \frac{1}{tan}.$

Notice that the *sine* and *cosine*, when obtained from the formulæ of $\alpha + \beta$ and $\alpha - \beta$, have the correct signs, and that therefore it is better to find the *cosine* thus, than by the formula $cos = \sqrt{1 - sin^2}$, which, giving the double sign, ±, leaves it ambiguous which is correct.

(20.) From the ratios of 30° and 45°, find those of 75° and − 15°.

By − 15° is meant an angle of 15° measured the contrary way to + 15°. The one angle, 15°, is in the first quadrant; the other, − 15°, is in the 4th quadrant.

Then, $sin\ 75^\circ = 45^\circ\ cos\ 30^\circ + cos\ 45^\circ\ sin\ 30^\circ.$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2},$$

$$= \frac{\sqrt{3}}{\sqrt{8}} + \frac{1}{\sqrt{8}},$$

$$= \frac{\sqrt{3} + 1}{\sqrt{8}}.$$

$$cos\ 75^\circ = cos\ 45^\circ\ cos\ 30^\circ - sin\ 45^\circ\ sin\ 30^\circ,$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2},$$

$$= \frac{\sqrt{3}}{\sqrt{8}} - \frac{1}{\sqrt{8}},$$

$$= \frac{\sqrt{3} - 1}{\sqrt{8}}.$$

$$
\begin{aligned}
\sin -15^\circ &= \sin(30^\circ - 45^\circ),\\
&= \sin 30^\circ \cos 45^\circ - \cos 30^\circ \sin 45^\circ,\\
&= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}},\\
&= \frac{1}{\sqrt{8}} - \frac{\sqrt{3}}{\sqrt{8}},\\
&= \frac{1-\sqrt{3}}{\sqrt{8}},\\
&= \frac{-(\sqrt{3}-1)}{\sqrt{8}}.
\end{aligned}
$$

$$
\begin{aligned}
\cos -15^\circ &= \cos(30^\circ - 45^\circ),\\
&= \cos 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ,\\
&= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}},\\
&= \frac{\sqrt{3}}{\sqrt{8}} + \frac{1}{\sqrt{8}},\\
&= \frac{1+\sqrt{3}}{\sqrt{8}}.
\end{aligned}
$$

Notice that the ratios of 15° equal, in magnitude, the ratios of − 15°, but that, while all the ratios of 15° are + (as being in the first quadrant) those of − 15° (in the fourth quadrant) are all − but the *cos* and *sec*.

Thus,

$$
\begin{aligned}
\tan &= \frac{\sin}{\cos},\\
&= \frac{\;\frac{-(\sqrt{3}-1)}{\sqrt{8}}\;}{\;\frac{1+\sqrt{3}}{\sqrt{8}}\;},\\
&= \frac{-(\sqrt{3}-1)}{1+\sqrt{3}}.
\end{aligned}
$$

(21.) If the diameter of a circle be 10 feet, what angle does an arc of 7 feet subtend?

Let d = diameter, c = circumference, A = the required angle.

Then, $$d = 10;$$
$$\therefore c = 10 + 3\cdot1415,$$
$$= 31\cdot415.$$

Then, the angle having an arc of 7 feet is to 360° (4 right angles) as 7 feet is 31·415 feet.

$$\therefore \text{A} : 360^\circ :: 7 : 31\cdot415;$$
$$\therefore \text{A} = \frac{360^\circ \times 7}{31\cdot415},$$
$$= 80^\circ\ 12'\ 59''.$$

(22.) If the radius be 8 feet, what is the length of the arc subtending an angle of 49°?

If $r = 8$, then $d = 16$,

and $c = 16 \times 3\cdot1415$,
$$= 50\cdot264.$$

Then, $$\text{arc} : 50\cdot264 :: 49^\circ : 360^\circ;$$
$$\therefore \text{arc} = \frac{50\cdot264 \times 49}{360},$$
$$= 6\cdot84 \text{ feet}.$$

(23.) If the length of an arc be 8 inches, and the angle 4°, what is the length of the radius?

Now, $$4^\circ : 360^\circ :: 8 : c;$$
$$\therefore c = \frac{360 \times 8}{4},$$
$$= 720;$$

then, $$r = \frac{c}{2 \times 3\cdot1415},$$

$$= \frac{720}{6 \cdot 283},$$
$$= 114 \cdot 5.$$

(24.) In a circle of 12 feet radius, find the number of degrees in an arc of 40 inches, and the length of an arc of 40°.

If $r = 12$, then $d = 24$.

Firstly: and $c = 24 \times 3 \cdot 1415 = 75 \cdot 4$.

Then, $3 \cdot 3 : 75 \cdot 4 :: \text{arc} : 360°$;

$$\because \text{arc} = \frac{360° \times 3 \cdot 3}{75 \cdot 4},$$
$$= 15° \, 45',$$

and therefore an arc of 40 inches (3·3 feet) subtends an angle of 15° 45′.

Secondly: as $360° : 40° :: 75 \cdot 4 : \text{arc}$;

$$\therefore \text{arc} = \frac{75 \cdot 4 \times 40}{360},$$
$$= 8 \cdot 38;$$

$\therefore$ an arc of $40° = 8 \cdot 38$ feet in length.

(25.) The circumference of a circle is 18 feet; find the length of an arc of 80°, and the number of degrees in an arc of 4 feet.

Firstly: as $360° : 80 :: 18 : \text{arc}$;

$$\therefore \text{arc} = \frac{80 \times 18}{360} = 4;$$

$\therefore$ arc of $80° = 4$ feet.

Secondly: the number of degrees in an—

$$\text{arc of 9 feet} = \frac{360° \times 9}{18},$$

$$= 180^\circ;$$

$$\therefore \text{arc of 9 feet} = 180^\circ.$$

(26.) In each of two circles, one of 3 inch radius, and the other of 4 feet, take an arc of 20° and compare the lengths.

$$c = r \times 2 \times 3{\cdot}1415,$$

$$= r \times 6{\cdot}2830;$$

then, $\text{one arc} = \dfrac{3 \times 6{\cdot}283}{18}$ inches,

and the other $= \dfrac{48 \times 6{\cdot}283}{18}$ inches,

then the lengths are respectively 1·047 and 16·752,

and their ratio $= \dfrac{1{\cdot}047}{16{\cdot}752}.$

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